GCSE ONLINE TUTORING

GEOMETRY & MEASURES / TOPIC GUIDE 50

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Further trigonometry

Higher-level GCSE material

Solve non-right triangles with the sine and cosine rules, calculate area using an included angle, interpret trigonometric graphs and identify right triangles inside 3D figures. Include the possible ambiguity when a sine gives two angles.

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Match opposite pairs and included angles carefully.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Your learning guide

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Independent educational resource

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Before you begin

Use right-triangle trigonometry, rearrange equations and work in degree mode.

Your route through this notebook

1. The sine rule and matching opposite pairs
2. The cosine rule for sides and angles
3. Triangle area using an included angle
4. Sine, cosine and tangent graphs
5. 3D angles, projections and connected triangles

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

The sine rule and matching opposite pairs

For a triangle with side a opposite angle A, and similarly b,B and c,C, the sine rule is a/sinA = b/sinB = c/sinC. A complete opposite side-angle pair is needed to determine another side or angle. Match labels by opposition rather than their position in a sketch.

When solving sinB = s with 0 < s < 1, B can be the acute calculator angle or its supplement 180° − B. In a triangle, retain only candidates that leave a positive third angle and satisfy the supplied geometry. This is the ambiguous SSA case. A side calculation with two known angles avoids that ambiguity.

A reliable method

1. Label each side opposite its angle.

2. Select an equality with one complete known pair.

3. For an unknown angle, test both sine candidates against the triangle total.

Worked example 1

A = 30°,B = 45°,a = 6 cm. Find b exactly.

STEP-BY-STEP SOLUTION

b/sin45° = 6/sin30°, so b = 6(√2/2)/(1/2) = 6√2 cm.

Worked example 2

A = 30°,a = 5,b = 8. Find possible B values to 1 decimal place.

STEP-BY-STEP SOLUTION

sinB = 8sin30°/5 = 0.8. B ≈ 53.1° or 126.9°. Both leave a positive third angle, so both are possible without more information.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

The inverse sine key returns one principal angle. It does not automatically rule out an obtuse triangle angle with the same sine.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 A = 30°,B = 90°,a = 4. Find b.

1.2 A = 45°,B = 30°,a = 10. Find b exactly.

1.3 If sinB = 0.5, state both candidates between 0° and 180°.

1.4 A = 50° and candidate B = 150°. Can both be triangle angles?

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

The cosine rule for sides and angles

The cosine rule is $c^2 = a^2 + b^2 - 2abcosC$, where C is the included angle between a and b and lies opposite c. Use it for two sides and their included angle, or three sides when seeking an angle. At $C = 90°$, $cosC = 0$, so the rule reduces to Pythagoras.

To find an angle, rearrange $cosC = (a^2 + b^2 - c^2)/(2ab)$, then apply inverse cosine. Choose c as the side opposite the desired angle. A negative cosine gives an obtuse angle and is not an error. A cosine value outside $[-1,1]$ indicates inconsistent data or a calculation mistake for a real triangle.

A reliable method

1. Choose the required side or angle and its opposite partner.

2. Use the included-angle structure in the formula.

3. Keep intermediate precision and check longest side versus largest angle.

Worked example 1

Find c for $a = 7, b = 8, C = 60°$.

STEP-BY-STEP SOLUTION

$c^2 = 49 + 64 - 112 \times 1/2 = 57$, so $c = \sqrt{57} \approx 7.55$.

Worked example 2

Find the angle opposite side 9 in a triangle with other sides 7 and 8.

STEP-BY-STEP SOLUTION

$cosC = (49 + 64 - 81)/(112) = 2/7$. $C \approx 73.4°$.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

The minus sign is part of the cosine rule. For an obtuse included angle, cosine is negative, so subtracting its product increases c^2.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Find c for a = 5,b = 5,C = 60°.

2.2 Find c for a = 6,b = 8,C = 90°.

2.3 Find C for sides a = 5,b = 5,c = 5.

2.4 If C = 120°,a = b = 4, find c exactly.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Triangle area using an included angle

A triangle’s area is ab sinC/2 for two sides a,b and their included angle C. The formula follows by drawing a perpendicular height h = bsinC and using area ah/2. It works for acute and obtuse included angles because sine remains positive between 0° and 180°.

If area and two sides are known, rearrange sinC = 2A/(ab). As with the sine rule, there may be two supplementary angle possibilities. Additional side information, a stated acute or obtuse condition, or another geometric restriction may select one. Do not confuse this area formula with a sector area formula; the same central angle may appear in both when calculating a circular segment.

A reliable method

1. Use the angle between the two supplied side lengths.

2. Substitute into ab sinC/2 or rearrange for sinC.

3. For a recovered angle, consider both supplementary possibilities.

Worked example 1

Find area for sides 7 cm and 10 cm with included angle 30°.

STEP-BY-STEP SOLUTION

Area = 7 × 10 × sin30°/2 = 17.5 cm².

Worked example 2

A triangle has sides 8 and 10 enclosing angle C and area 20. Find possible C.

STEP-BY-STEP SOLUTION

sinC = 40/80 = 1/2, giving C = 30° or 150°. Both have the same area with these two side lengths.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

An angle not between the known sides cannot be inserted into this formula directly. First find a suitable included angle or another needed side.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Find area for sides 6,9 and included angle 90°.

3.2 Find exact area for sides 4,8 and included angle 60°.

3.3 Find possible included angles for sides 6,8 and area 12.

3.4 Sides 5 and 7 enclose an angle. What is the maximum possible triangle area?

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 4 / UNDERSTAND

Sine, cosine and tangent graphs

With angles in degrees, sinx and cosx have period 360° and values between −1 and 1. Sine passes through (0,0), peaks at (90,1), returns to zero at 180° and reaches −1 at 270°. Cosine begins at (0,1) and crosses zero at 90° and 270°.

Tangent has period 180°, crosses zero at multiples of 180° and is undefined at 90° + 180°k for integer k. Separate its branches at these vertical asymptotes. A trigonometric equation on a specified interval may have several solutions. Use symmetry and periodicity, then check every candidate lies within the stated inclusive or exclusive endpoints.

A reliable method

1. Use degree axes and mark key points or asymptotes.

2. Find principal solutions, then use symmetry and period.

3. List all solutions within the exact requested interval.

Worked example 1

Solve sinx = 1/2 for 0° ≤ x ≤ 360°.

STEP-BY-STEP SOLUTION

The acute angle is 30°. Sine is also positive in quadrant II: x = 150°. Solutions 30°,150°.

Worked example 2

Solve cosx = 0 for 0° ≤ x ≤ 360°.

STEP-BY-STEP SOLUTION

Cosine crosses the axis at 90° and 270°. Both are in the interval.

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LESSON 4 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Do not join a tangent curve across an asymptote. Undefined points are not very large finite graph values.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

4.1 State the period of sinx in degrees.

4.2 Solve tanx = 1 for $0° \leq x \leq 360°$.

4.3 Solve cosx = 1/2 for $0° \leq x \leq 360°$.

4.4 State tan90°.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 5 / UNDERSTAND

3D angles, projections and connected triangles

An angle between a line and a plane is measured between the line and its perpendicular projection onto the plane. In a cuboid, the projection of a space diagonal onto the base is the base diagonal. This creates a right triangle using the vertical height, the base diagonal and the space diagonal.

Find a face diagonal with Pythagoras before using trigonometry for the required 3D angle. Other angles, such as between a space diagonal and a vertical edge, use different sides of the same triangle and are complementary when appropriate. Draw the triangle separately with clear labels; a perspective drawing’s apparent angle is not a measurement of the true spatial angle.

A reliable method

1. Identify the plane, line and its projection.

2. Calculate the projected length using the appropriate face.

3. Use a right-triangle ratio and retain unrounded intermediate values.

Worked example 1

A cuboid has base 3 by 4 and height 5. Find the angle its space diagonal makes with the base plane.

STEP-BY-STEP SOLUTION

Base diagonal $\sqrt{(9 + 16)} = 5$. Thus $\tan\theta = 5/5 = 1$, so $\theta = 45°$.

Worked example 2

A square-based pyramid has base side 6 and height $3\sqrt{2}$. Find angle from a sloping edge to the base plane.

STEP-BY-STEP SOLUTION

Projection from centre to base corner is half the base diagonal: $3\sqrt{2}$. $\tan\theta = 3\sqrt{2}/(3\sqrt{2}) = 1$, so $\theta = 45°$.

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LESSON 5 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

For a sloping edge of a square pyramid use centre-to-corner distance, not centre-to-edge-midpoint distance. The latter is used for a face’s slant height.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

5.1 A cuboid has base diagonal 12 and height 5. Express angle to base plane.

5.2 A cuboid has base 6 by 8 and height 10. Find diagonal-to-base angle.

5.3 A line has angle 35° to a horizontal plane. Find its angle to the vertical direction in the same perpendicular cross-section.

5.4 A cube has side $s > 0$. Express the space-diagonal angle to a face plane.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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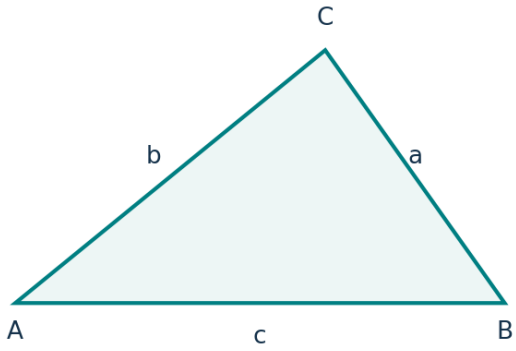
VISUAL EXPLAINER / RELATIONSHIPS, NOT GUESSWORK

Read the geometry • 1

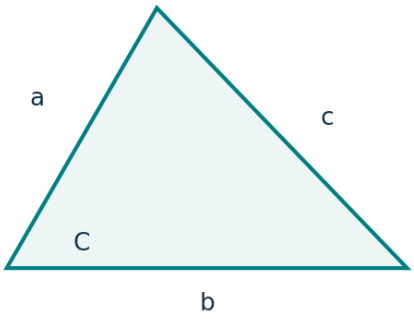
Match the angle with its opposite or included sides

Sine rule: opposite pairs



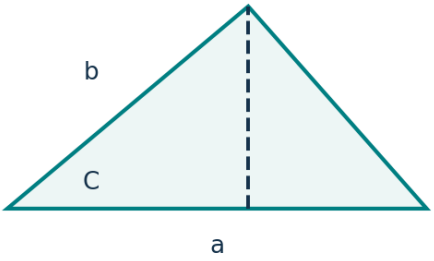
a/sin A = b/sin B

Cosine rule: included angle



$c^2 = a^2 + b^2 - 2ab \cos C$

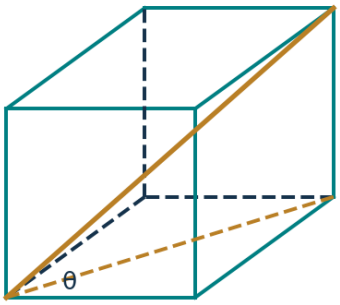
Area using an included angle



A = ab sin C / 2

Angle between a line and a plane

tan θ = height / projection



Projection: base diagonal

WHAT THE FIGURES ESTABLISH

In the sine rule, each side is opposite its same-letter angle. In the cosine rule and triangle-area formula, the known angle is between the two multiplying side lengths. A line-to-plane angle uses the line's perpendicular projection into the plane, not its apparent angle on the page. These structural diagrams are not to scale.

Discuss: identify the given information, the geometric condition and the conclusion. Recalculate a labelled result or explain a theorem without measuring the sketch. Unless explicitly stated, diagrams are illustrative and are not measurement questions.

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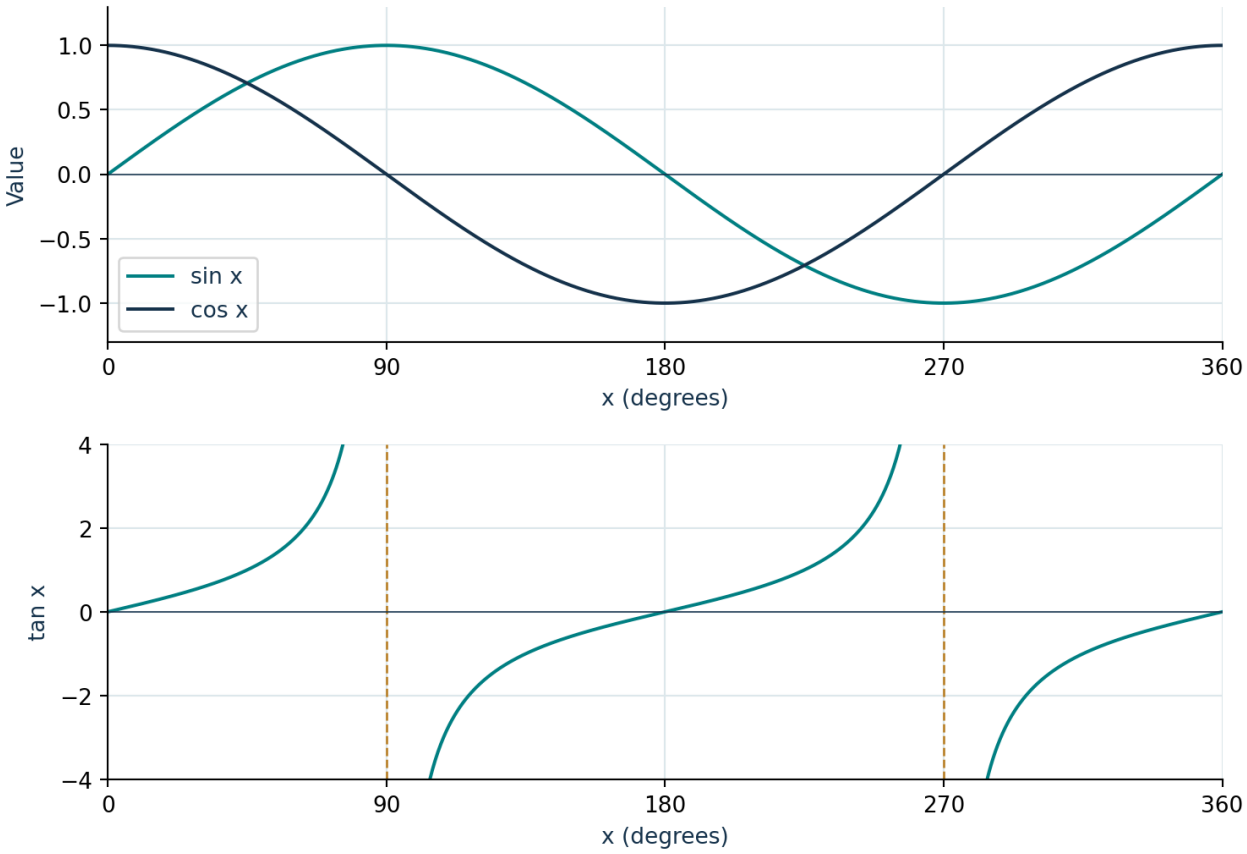
VISUAL EXPLAINER / RELATIONSHIPS, NOT GUESSWORK

Read the geometry • 2

Trigonometric graphs: angles in degrees



WHAT THE FIGURES ESTABLISH

Sine and cosine repeat every 360° and stay between −1 and 1. Tangent repeats every 180° and has separate branches divided by asymptotes at 90° and 270° in this interval. Do not draw connecting segments across those undefined points. Use symmetry and period to list every solution in a specified interval.

Discuss: identify the given information, the geometric condition and the conclusion. Recalculate a labelled result or explain a theorem without measuring the sketch. Unless explicitly stated, diagrams are illustrative and are not measurement questions.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. A = 30°,B = 45°,a = 8. Find b exactly. [3 marks]

R2. A = 30°,a = 6,b = 9. Find possible B values to 1 decimal place. [3 marks]

R3. Find c for a = 5,b = 7,C = 60°. [3 marks]

R4. Find angle opposite side 7 in a 7–7–7 triangle using cosine rule. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R5. Find exact triangle area for sides 10 and 12 with included angle 60°. [2 marks]

R6. Solve $\sin x = 0$ for $0° \leq x \leq 360°$. [2 marks]

R7. Solve $\tan x = -1$ for $0° \leq x \leq 360°$. [2 marks]

R8. A cuboid has base 5 by 12 and height 13. Find space-diagonal angle to the base. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • The sine rule and matching opposite pairs

1.1 $A = 30°, B = 90°, a = 4$. Find b.

$b = 4/\sin 30° = 8$.

1.2 $A = 45°, B = 30°, a = 10$. Find b exactly.

$10\sin 30°/\sin 45° = 5\sqrt{2}$.

1.3 If $\sin B = 0.5$, state both candidates between 0° and 180°.

30° and 150°.

1.4 $A = 50°$ and candidate $B = 150°$. Can both be triangle angles?

No, their sum exceeds 180°.

Lesson 2 • The cosine rule for sides and angles

2.1 Find c for $a = 5, b = 5, C = 60°$.

$c^2 = 25$, so $c = 5$.

2.2 Find c for $a = 6, b = 8, C = 90°$.

$c = 10$.

2.3 Find C for sides $a = 5, b = 5, c = 5$.

$\cos C = 1/2$, so $C = 60°$.

2.4 If $C = 120°, a = b = 4$, find c exactly.

$c^2 = 16 + 16 - 32(-1/2) = 48$; $c = 4\sqrt{3}$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Triangle area using an included angle

3.1 Find area for sides 6,9 and included angle 90°.

27 square units.

3.2 Find exact area for sides 4,8 and included angle 60°.

8√3 square units.

3.3 Find possible included angles for sides 6,8 and area 12.

sinC = 24/48 = 1/2: 30° or 150°.

3.4 Sides 5 and 7 enclose an angle. What is the maximum possible triangle area?

ab/2 = 17.5 square units, achieved at 90°.

Lesson 4 • Sine, cosine and tangent graphs

4.1 State the period of sinx in degrees.

360°.

4.2 Solve tanx = 1 for 0° ≤ x ≤ 360°.

45° and 225°.

4.3 Solve cosx = 1/2 for 0° ≤ x ≤ 360°.

60° and 300°.

4.4 State tan90°.

Undefined.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 5 • 3D angles, projections and connected triangles

5.1 A cuboid has base diagonal 12 and height 5. Express angle to base plane.

$\theta = \tan^{-1}(5/12) \approx 22.6°$.

5.2 A cuboid has base 6 by 8 and height 10. Find diagonal-to-base angle.

Base diagonal 10; angle 45°.

5.3 A line has angle 35° to a horizontal plane. Find its angle to the vertical direction in the same perpendicular cross-section.

55°.

5.4 A cube has side $s > 0$. Express the space-diagonal angle to a face plane.

Base diagonal $s\sqrt{2}$; $\tan\theta = s/(s\sqrt{2}) = 1/\sqrt{2}$, so $\theta \approx 35.3°$, independent of size.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. A = 30°,B = 45°,a = 8. Find b exactly. [3 marks]

b = 8sin45°/sin30° = 8√2. [1 for opposite pairs; 1 for sine rule; 1 for result.]

R2. A = 30°,a = 6,b = 9. Find possible B values to 1 decimal place. [3 marks]

sinB = 0.75, so B ≈ 48.6° or 131.4°; both fit with A. [1 for sine; 1 for first angle; 1 for supplement/check.]

R3. Find c for a = 5,b = 7,C = 60°. [3 marks]

$c^2 = 25 + 49 - 70/2 = 39$, so c = √39. [1 for rule; 1 for substitution; 1 for root.]

R4. Find angle opposite side 7 in a 7–7–7 triangle using cosine rule. [3 marks]

cosC = (49 + 49 − 49)/98 = 1/2; C = 60°. [1 for setup; 1 for cosine; 1 for angle.]

R5. Find exact triangle area for sides 10 and 12 with included angle 60°. [2 marks]

60sin60° = 30√3. [1 for formula; 1 for exact result.]

R6. Solve sinx = 0 for 0° ≤ x ≤ 360°. [2 marks]

0°,180°,360°. [1 for principal/period idea; 1 for complete list.]

R7. Solve tanx = −1 for 0° ≤ x ≤ 360°. [2 marks]

135°,315°. [1 for reference angle; 1 for full list.]

R8. A cuboid has base 5 by 12 and height 13. Find space-diagonal angle to the base. [3 marks]

Base diagonal 13; tanθ = 13/13 = 1; θ = 45°. [1 for diagonal; 1 for ratio; 1 for angle.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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