GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 11

11

Algebra basics and operations with terms

Shared GCSE foundations

Letters let us describe many numbers at once. Learn the language of algebra, combine terms correctly and distinguish expressions from equations. The aim is to explain every change, not just memorise rules.

3x + 5x = 8x

Like terms combine; products follow a different rule.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

INDEPENDENT GCSE MATHS • WEBSITE EDITION • NOVEMBER 2026

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Before you begin

Work confidently with signed numbers, multiplication, division and powers.

Your route through this notebook

1. Reading algebra and identifying terms
2. Collecting like terms
3. Multiplying and dividing algebraic terms

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Reading algebra and identifying terms

A variable represents a number that may vary or be unknown. In 5x + 3, the coefficient 5 multiplies x and the constant 3 is added. A term is a part of an expression separated by addition or subtraction at the outermost level. Thus 4x − 7y + 2 has terms 4x, −7y and 2; the minus belongs to the term.

Algebra normally omits the multiplication sign: 3ab means 3 × a × b. A power is different: a^2 means a × a, whereas 2a means a + a. An expression has no assertion of equality; an equation states that two expressions are equal for particular values. An identity is true for every permitted value of its variables.

A reliable method

1. Read adjacent letters and numbers as multiplication.

2. Identify terms together with their signs.

3. Decide whether you are simplifying an expression or solving an equation.

Worked example 1

Identify the terms and coefficients in 6p − 3q + 8.

STEP-BY-STEP SOLUTION

The terms are 6p, −3q and 8. The coefficients of p and q are 6 and −3. The constant is 8.

Worked example 2

Write a number n increased by 7, and seven times n.

STEP-BY-STEP SOLUTION

Increased by 7 is n + 7. Seven times n is 7n. These describe different operations.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A letter is a number, not an object label. If a is the number of apples, 3a counts three times that number. Two different letters can have the same value; different symbols do not guarantee different numbers.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 List the terms of $2x - 5y - 9$.

1.2 Write $4 \times a \times b$ without multiplication signs.

1.3 Explain the difference between x^2 and $2x$.

1.4 Classify $3x + 2$ and $3x + 2 = 14$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Collecting like terms

Like terms have exactly the same variable part, including powers. Because 3x and 5x count the same kind of quantity, their coefficients can be added: 3x + 5x = 8x. The terms x and x^2 are unlike; they represent different expressions. Constants can be collected with constants.

Subtraction is addition of a negative term, so carry every sign while rearranging. Commutativity allows terms in a sum to be regrouped, but changing a sign changes the value. A simplified expression should have each type of term collected and no unnecessary multiplication signs. You are not finding a numerical answer unless the variables have specified values.

A reliable method

1. Group terms with matching letters and matching powers.

2. Add their signed coefficients.

3. Keep unlike terms separate and combine constants.

Worked example 1

Simplify $7x - 3 + 2x + 8$.

STEP-BY-STEP SOLUTION

The x terms give $7x + 2x = 9x$. Constants give $-3 + 8 = 5$. Result: $9x + 5$.

Worked example 2

Simplify $4a^2 + 3a - a^2 - 5a$.

STEP-BY-STEP SOLUTION

Square terms give $3a^2$; linear terms give $-2a$. Result: $3a^2 - 2a$.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

3x + 4y cannot generally become 7xy. Test x = 1 and y = 2: the sum is 11 but 7xy is 14. A single counterexample disproves the suggested rule.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Simplify $8x - 3x + x$.

2.2 Simplify $5a + 2b - a + 7b$.

2.3 Simplify $6p^2 - p + 2p^2 + 4p$.

2.4 Simplify $9 - 4y - 12 + y$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Multiplying and dividing algebraic terms

A product combines factors rather than adding coefficients. Multiply numerical coefficients and then combine the variable factors. For example, $(3x)(4x) = 12x^2$ because $x \times x = x^2$. Different letters remain as separate factors: $x \times y = xy$. Sign rules are the same as for ordinary numbers.

Division can be written as a fraction and simplified by cancelling common non-zero factors. A cancelled variable cannot be zero in the original denominator. When dividing powers with the same base, subtract their indices. Do not cancel across addition: $(x + 3)/x$ is not 3 because x is not a factor of the whole numerator.

A reliable method

1. Multiply or divide coefficients with their signs.
2. Combine matching variable factors using index laws.
3. State restrictions caused by any original denominator.

Worked example 1

Simplify $(-3a)(2a^2b)$.

STEP-BY-STEP SOLUTION

The coefficients give -6; $a \times a^2 = a^3$. Therefore $-6a^3b$.

Worked example 2

Simplify $18x^3y \div 6xy$, where x and y are non-zero.

STEP-BY-STEP SOLUTION

Divide coefficients: $18/6 = 3$. Cancel y and one x, leaving x^2. Result: $3x^2$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Addition and multiplication follow different rules: $3x + 4x = 7x$, but $(3x)(4x) = 12x^2$. Identify the operation before changing coefficients or powers.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Simplify $(5x)(-2y)$.

3.2 Simplify $(3a^2)(4a^3)$.

3.3 Simplify $20m^3n^2 \div 5mn$, for non-zero m,n.

3.4 A rectangle has sides 3x cm and 5x cm. Express its perimeter and area.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Simplify $9x - 4y - 3x + y$. [2 marks]

R2. Simplify $5a^2 + 2a - 3a^2 + 7a$. [2 marks]

R3. Simplify $(-4p^2)(3pq)$. [2 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Simplify $24x^4y^2 \div 8x^2y$, with non-zero x,y. [2 marks]

R5. Explain why $2x + 3x^2$ is not $5x^3$. [2 marks]

R6. A rectangle has sides 2a and 7a. Find expressions for area and perimeter. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Reading algebra and identifying terms

1.1 List the terms of $2x - 5y - 9$.

$2x$, $-5y$ and -9.

1.2 Write $4 \times a \times b$ without multiplication signs.

$4ab$.

1.3 Explain the difference between x^2 and $2x$.

x^2 is x multiplied by x; $2x$ is x added to itself. For $x = 3$ they give 9 and 6.

1.4 Classify $3x + 2$ and $3x + 2 = 14$.

$3x + 2$ is an expression. $3x + 2 = 14$ is an equation.

Lesson 2 • Collecting like terms

2.1 Simplify $8x - 3x + x$.

$(8 - 3 + 1)x = 6x$.

2.2 Simplify $5a + 2b - a + 7b$.

$4a + 9b$.

2.3 Simplify $6p^2 - p + 2p^2 + 4p$.

$8p^2 + 3p$.

2.4 Simplify $9 - 4y - 12 + y$.

$-3y - 3$, or $-3 - 3y$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Multiplying and dividing algebraic terms

3.1 Simplify $(5x)(-2y)$.

$-10xy$.

3.2 Simplify $(3a^2)(4a^3)$.

$12a^5$.

3.3 Simplify $20m^3n^2 \div 5mn$, for non-zero m,n.

$4m^2n$.

3.4 A rectangle has sides 3x cm and 5x cm. Express its perimeter and area.

Perimeter = $2(3x + 5x) = 16x$ cm. Area = $(3x)(5x) = 15x^2$ cm². For lengths, $x > 0$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Simplify $9x - 4y - 3x + y$. [2 marks]

$6x - 3y$. [1 for each collected variable.]

R2. Simplify $5a^2 + 2a - 3a^2 + 7a$. [2 marks]

$2a^2 + 9a$. [1 for each term type.]

R3. Simplify $(-4p^2)(3pq)$. [2 marks]

$-12p^3q$. [1 for coefficient; 1 for variables.]

R4. Simplify $24x^4y^2 \div 8x^2y$, with non-zero x,y. [2 marks]

$3x^2y$. [1 for coefficient; 1 for cancelled powers.]

R5. Explain why $2x + 3x^2$ is not $5x^3$. [2 marks]

The terms have different powers and cannot be collected. For $x = 2$ the original is 16, while $5x^3 = 40$. [1 for unlike terms; 1 for valid illustration.]

R6. A rectangle has sides 2a and 7a. Find expressions for area and perimeter. [3 marks]

Area $= 14a^2$; perimeter $= 2(2a + 7a) = 18a$. [1 for each formula; 1 for correct simplification.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 12

12

Substitution and writing expressions

Shared GCSE foundations

Turn words into mathematics and evaluate expressions accurately. Careful substitution includes brackets around negative values and respects the original order of operations. Use expressions to model situations while keeping units and assumptions visible.

$$3(-4)^2 - 2(-4) = 56$$

Substitute in brackets, then follow operation order.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

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Before you begin

Understand terms, coefficients, signed arithmetic and order of operations.

Your route through this notebook

1. Substitution with positive and negative inputs
2. Translating words into expressions
3. Building models from contexts

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Substitution with positive and negative inputs

Substitution replaces every occurrence of a variable by its assigned value. The structure of the expression stays the same: powers, brackets and operations must still be applied in the correct order. If a value is negative, enclose it in brackets to show exactly what is being substituted.

A coefficient multiplies the substituted value. In $4x^2$, square x first and multiply by 4 afterwards. In $(4x)^2$, the whole product is squared. When several variables are present, identify each value before calculating. Keep fractions exact when that makes the calculation cleaner.

A reliable method

1. Copy the expression and replace each letter using brackets.

2. Evaluate powers and brackets before multiplication and addition.

3. Check signs and whether the resulting size is plausible.

Worked example 1

Evaluate $3x^2 - 2x$ for $x = -4$.

STEP-BY-STEP SOLUTION

$3(-4)^2 - 2(-4) = 3 \times 16 + 8 = 56$.

Worked example 2

Evaluate $(a + 2b)/c$ when $a = 5$, $b = -3$, $c = 2$.

STEP-BY-STEP SOLUTION

The numerator is $5 + 2(-3) = -1$. Divide the whole numerator by 2: $-1/2$.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

For $x = -3$, $x^2 = 9$ but $-x^2 = -9$. The minus in the input belongs inside the substitution brackets; a minus outside the original square remains outside.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Evaluate $5x + 7$ for $x = 6$.

1.2 Evaluate $y^2 + 4y$ for $y = -5$.

1.3 Evaluate $2ab - b^2$ for $a = -2$, $b = 3$.

1.4 Evaluate $6t^2$ for $t = 1/2$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Translating words into expressions

An expression describes a calculation without necessarily giving an answer. Choose a variable for the changing quantity, then translate the operations in their stated order. “Five more than twice n” is 2n + 5, while “twice the sum of n and five” is 2(n + 5). Brackets preserve grouped instructions.

Subtraction and division are sensitive to order. “Seven less than n” is n − 7, and “n divided by four” is n/4. Test a small input in your proposed expression and compare it with the verbal instructions. This catches many translation errors before any equation is formed.

A reliable method

1. Define the variable in words.

2. Identify the order and any grouped quantity.

3. Test the expression using a simple input.

Worked example 1

Write an expression for three less than four times k.

STEP-BY-STEP SOLUTION

Four times k is 4k; three less is 4k − 3.

Worked example 2

A number n is increased by 6, then the result is divided by 4. Write an expression.

STEP-BY-STEP SOLUTION

The whole increased amount is divided: (n + 6)/4. Without brackets, n + 6/4 describes a different operation.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

The order of words can differ from the order of symbols. “Subtract p from 10” means start with 10 and remove p: 10 − p. Use a numerical test rather than following word order blindly.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Write five more than three times x.

2.2 Write half the sum of a and b.

2.3 Write the square of the difference between n and 4.

2.4 Write the result of subtracting y from 12 and multiplying by 3.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Building models from contexts

A mathematical model represents selected features of a situation. A fixed charge is a constant; a cost per item produces a variable term. If a taxi charges £4 plus £1.80 per kilometre, its model is C = 4 + 1.8d, with d in kilometres and C in pounds. The model assumes the same rate applies throughout and excludes additional charges.

Define units and valid inputs. A number of tickets should be a non-negative integer, whereas a distance can usually vary continuously. An expression can calculate the output; an equation can be formed when the output is known. Interpreting the answer is part of the mathematics, not an optional final sentence.

A reliable method

1. Separate fixed quantities from quantities that vary.

2. Define each variable and its unit.

3. State important assumptions and check whether the input is realistic.

Worked example 1

A delivery costs £6 plus £2.50 per parcel. Model the cost and find it for 8 parcels.

STEP-BY-STEP SOLUTION

Let n be the number of parcels. C = 6 + 2.5n pounds. For n = 8, C = 6 + 20 = £26.

Worked example 2

A rectangle has width x and length x + 3. Write its perimeter.

STEP-BY-STEP SOLUTION

P = 2x + 2(x + 3) = 4x + 6. If measurements are in cm, P is in cm and x must be positive.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A model can give a numerical output for an invalid input. A ticket-cost formula might accept n = 2.5 algebraically, but half a ticket may not be sold. The domain comes from the situation.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 A gym charges £15 joining fee and £8 per visit. Model total cost for v visits.

3.2 Use that model for 7 visits.

3.3 A rectangle has sides x and x + 5. Write its area.

3.4 A tank starts with 90 litres and loses 4 litres per minute. Model its content and state the valid time interval until empty.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Evaluate $2x^2 - 5x + 1$ for $x = -3$. [3 marks]

R2. Evaluate $(3a - b)/(a + b)$ for $a = 4$, $b = -1$. [3 marks]

R3. Write three times the sum of n and 8. [1 mark]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Write five less than the square of p. [1 mark]

R5. A hire company charges £12 plus £7 per hour. Model the cost and calculate 5 hours. [3 marks]

R6. A rectangle has sides x + 2 and x + 7. Write its perimeter in simplest form. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Substitution with positive and negative inputs

1.1 Evaluate $5x + 7$ for $x = 6$.

$5 \times 6 + 7 = 37$.

1.2 Evaluate $y^2 + 4y$ for $y = -5$.

$25 - 20 = 5$.

1.3 Evaluate $2ab - b^2$ for $a = -2$, $b = 3$.

$2(-2)(3) - 9 = -12 - 9 = -21$.

1.4 Evaluate $6t^2$ for $t = 1/2$.

$6 \times (1/2)^2 = 6/4 = 3/2$.

Lesson 2 • Translating words into expressions

2.1 Write five more than three times x.

$3x + 5$.

2.2 Write half the sum of a and b.

$(a + b)/2$.

2.3 Write the square of the difference between n and 4.

$(n - 4)^2$.

2.4 Write the result of subtracting y from 12 and multiplying by 3.

$3(12 - y)$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Building models from contexts

3.1 A gym charges £15 joining fee and £8 per visit. Model total cost for v visits.

C = 15 + 8v, with v a non-negative integer and C in pounds.

3.2 Use that model for 7 visits.

C = 15 + 8 × 7 = £71.

3.3 A rectangle has sides x and x + 5. Write its area.

A = x(x + 5), or $x^2 + 5x$. For lengths, x > 0.

3.4 A tank starts with 90 litres and loses 4 litres per minute. Model its content and state the valid time interval until empty.

V = 90 − 4t litres. For this constant-rate model, 0 ≤ t ≤ 22.5 minutes.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Evaluate $2x^2 - 5x + 1$ for $x = -3$. [3 marks]

$2 \times 9 + 15 + 1 = 34$. [1 for substitution; 1 for signs/power; 1 for answer.]

R2. Evaluate $(3a - b)/(a + b)$ for $a = 4$, $b = -1$. [3 marks]

$(12 + 1)/(4 - 1) = 13/3$. [1 for substitution; 1 for numerator/denominator; 1 for quotient.]

R3. Write three times the sum of n and 8. [1 mark]

$3(n + 8)$. [1 for correct grouping.]

R4. Write five less than the square of p. [1 mark]

$p^2 - 5$. [1 for expression.]

R5. A hire company charges £12 plus £7 per hour. Model the cost and calculate 5 hours. [3 marks]

$C = 12 + 7h$; $C(5) = £47$. [1 for fixed term; 1 for hourly term; 1 for cost.]

R6. A rectangle has sides $x + 2$ and $x + 7$. Write its perimeter in simplest form. [3 marks]

$P = 2(x + 2) + 2(x + 7) = 4x + 18$. [1 for perimeter structure; 1 for expansion; 1 for collection.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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ALGEBRA & GRAPHS / TOPIC GUIDE 13

13

Expanding and factorising: foundations

Shared GCSE core

Expansion and factorisation are reverse processes. Learn to distribute multiplication over addition, remove common factors and connect simple quadratic factors to their expanded forms. Check transformations by reversing them.

$$x^2 + 8x + 15 = (x + 3)(x + 5)$$

Expansion and factorisation reverse each other.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

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Before you begin

Collect like terms, multiply algebraic terms and identify common factors.

Your route through this notebook

1. Expanding single brackets
2. Factorising out a common factor
3. Double brackets and simple quadratics

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

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LESSON 1 / UNDERSTAND

Expanding single brackets

A multiplier outside a bracket multiplies every term inside. This distributive property follows from splitting a quantity into parts: $3(x + 4) = 3x + 12$. A negative multiplier changes the signs of the products; it is safest to write each product explicitly before collecting terms.

If a bracket is preceded by a subtraction sign, treat the multiplier as -1. For example, $8 - (x - 2)$ becomes $8 - x + 2$. Expanding is not solving: there is no unknown value to find unless an equation is supplied. Equivalent expressions have the same value for every permitted input.

A reliable method

1. Multiply every term in the bracket.
2. Keep each term’s sign and the multiplier’s sign.
3. Collect like terms only after the bracket is removed.

Worked example 1

Expand $5(2x - 3)$.

STEP-BY-STEP SOLUTION

$5 \times 2x + 5 \times (-3) = 10x - 15$.

Worked example 2

Simplify $4x - 2(3x - 5)$.

STEP-BY-STEP SOLUTION

Expand the product as $-6x + 10$. Then $4x - 6x + 10 = -2x + 10$.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

3(x + 4) does not equal 3x + 4: the 4 must also be multiplied. Test x = 0 to see the difference immediately: 12 versus 4.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Expand 4(x + 7).

1.2 Expand −3(2a − 5).

1.3 Simplify 2(x + 3) + 5(x − 1).

1.4 Simplify 9 − (2y + 4).

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Factorising out a common factor

Factorising rewrites a sum as a product. It reverses expansion. In $6x + 15$, both terms contain a factor 3, so $6x + 15 = 3(2x + 5)$. The bracket records what remains after dividing each term by the chosen common factor.

To factorise fully, take the greatest common numerical factor and every variable factor shared by all terms, using the smallest shared power. A term with no variable limits which letters can be taken outside. Negative factors are sometimes useful, but any proposed factorisation should be checked by expanding it back.

A reliable method

1. Find the highest common factor of all terms.
2. Divide every term by that factor to form the bracket.
3. Expand the result to check that every original term returns.

Worked example 1

Factorise $12x^2 + 18x$ fully.

STEP-BY-STEP SOLUTION

The HCF is $6x$. Dividing gives $2x$ and 3, so $6x(2x + 3)$.

Worked example 2

Factorise $8a - 20$.

STEP-BY-STEP SOLUTION

The common numerical factor is 4. Result: $4(2a - 5)$.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Writing $12x^2 + 18x$ as $3x(4x + 6)$ is correct but not fully factorised because the bracket still has a factor 2. Full factorisation removes every common factor.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Factorise $10x + 25$.

2.2 Factorise $15y^2 - 5y$.

2.3 Factorise $14ab + 21a$.

2.4 Check whether $6(x + 4)$ factorises $6x + 4$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Double brackets and simple quadratics

To expand two brackets, every term in one bracket must multiply every term in the other. In $(x + a)(x + b)$, the products give $x^2 + ax + bx + ab$, which collect to $x^2 + (a + b)x + ab$. This connects factorisation to finding two numbers with a required sum and product.

For a monic quadratic $x^2 + bx + c$, seek two integers whose sum is b and whose product is c. If c is negative, the integers have opposite signs. If c is positive and b is negative, both integers are negative. Not every quadratic has integer factors, so a failed search does not justify inventing brackets.

A reliable method

1. For expansion, list all four products before collecting.

2. For factorisation, find the required sum and product.

3. Expand the factors to check both the middle and constant terms.

Worked example 1

Expand $(x + 4)(x - 3)$.

STEP-BY-STEP SOLUTION

Products: $x^2 - 3x + 4x - 12$. Collect: $x^2 + x - 12$.

Worked example 2

Factorise $x^2 - 7x + 12$.

STEP-BY-STEP SOLUTION

Find numbers with product 12 and sum -7: -3 and -4. Therefore $(x - 3)(x - 4)$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

$(x + 5)^2$ is not $x^2 + 25$. It is $(x + 5)(x + 5) = x^2 + 10x + 25$. The two middle products do not disappear.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Expand $(x + 2)(x + 7)$.

3.2 Expand $(x - 5)(x + 3)$.

3.3 Factorise $x^2 + 8x + 15$.

3.4 Factorise $x^2 - x - 12$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Expand $-4(3x - 2)$. [2 marks]

R2. Simplify $3(2x + 1) - 2(x - 4)$. [3 marks]

R3. Factorise $18a^2 - 12a$ fully. [2 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Expand $(x - 6)(x - 2)$. [3 marks]

R5. Factorise $x^2 + 2x - 24$. [2 marks]

R6. A rectangle has width $x + 3$ and length $x + 8$. Express its area in expanded form. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Expanding single brackets

1.1 Expand $4(x + 7)$.

$4x + 28$.

1.2 Expand $-3(2a - 5)$.

$-6a + 15$.

1.3 Simplify $2(x + 3) + 5(x - 1)$.

$2x + 6 + 5x - 5 = 7x + 1$.

1.4 Simplify $9 - (2y + 4)$.

$9 - 2y - 4 = 5 - 2y$.

Lesson 2 • Factorising out a common factor

2.1 Factorise $10x + 25$.

$5(2x + 5)$.

2.2 Factorise $15y^2 - 5y$.

$5y(3y - 1)$.

2.3 Factorise $14ab + 21a$.

$7a(2b + 3)$.

2.4 Check whether $6(x + 4)$ factorises $6x + 4$.

No. Expanding gives $6x + 24$. A correct factorisation is $2(3x + 2)$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Double brackets and simple quadratics

3.1 Expand $(x + 2)(x + 7)$.

$x^2 + 9x + 14$.

3.2 Expand $(x - 5)(x + 3)$.

$x^2 - 2x - 15$.

3.3 Factorise $x^2 + 8x + 15$.

$3 + 5 = 8$ and $3 \times 5 = 15$, so $(x + 3)(x + 5)$.

3.4 Factorise $x^2 - x - 12$.

$-4 + 3 = -1$ and $-4 \times 3 = -12$, so $(x - 4)(x + 3)$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Expand $-4(3x - 2)$. [2 marks]

$-12x + 8$. [1 for each product.]

R2. Simplify $3(2x + 1) - 2(x - 4)$. [3 marks]

$6x + 3 - 2x + 8 = 4x + 11$. [1 for each expansion; 1 for collection.]

R3. Factorise $18a^2 - 12a$ fully. [2 marks]

$6a(3a - 2)$. [1 for 6a; 1 for bracket.]

R4. Expand $(x - 6)(x - 2)$. [3 marks]

$x^2 - 2x - 6x + 12 = x^2 - 8x + 12$. [1 for four products; 1 for signs; 1 for collection.]

R5. Factorise $x^2 + 2x - 24$. [2 marks]

$(x + 6)(x - 4)$. [1 for suitable pair; 1 for factors.]

R6. A rectangle has width $x + 3$ and length $x + 8$. Express its area in expanded form. [3 marks]

$A = (x + 3)(x + 8) = x^2 + 11x + 24$. [1 for product model; 1 for expansion; 1 for simplification.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 14

14

Advanced expansion and factorisation

Higher-level GCSE material

Extend bracket methods to harder quadratics, differences of squares and products of three brackets. Choose a factorisation strategy by inspecting the structure first, then verify your answer by expansion.

$$a^2 - b^2 = (a - b)(a + b)$$

Look for structure before choosing a method.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Expand double brackets, collect powers and factorise monic quadratics.

Your route through this notebook

1. Quadratics with a leading coefficient
2. Difference of squares and complete factorisation
3. Three brackets and algebraic structure

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Quadratics with a leading coefficient

A quadratic $ax^2 + bx + c$ with $a \neq 1$ may factor into two linear brackets whose x coefficients multiply to a. A systematic method splits the middle term: find two numbers with product ac and sum b. Replace bx by those two terms, then factor by grouping.

The grouping works because each pair produces the same bracket. If the brackets do not match, check the selected numbers and signs. Always remove a common factor first. Integer factorisation is useful but not universal: some quadratics need other solving methods.

A reliable method

1. Take out any common factor.

2. Find a pair with product ac and sum b, then split bx.

3. Group into matching brackets and check by expansion.

Worked example 1

Factorise $6x^2 + 11x + 3$.

STEP-BY-STEP SOLUTION

ac = 18. Use 9 and 2: $6x^2 + 9x + 2x + 3 = 3x(2x + 3) + 1(2x + 3) = (3x + 1)(2x + 3)$.

Worked example 2

Factorise $2x^2 - 7x + 3$.

STEP-BY-STEP SOLUTION

ac = 6; use −6 and −1. Then $2x(x - 3) - 1(x - 3) = (2x - 1)(x - 3)$.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Finding numbers whose product is c alone works for monic quadratics, but not as a complete method when a ≠ 1. Check the coefficient of x^2 as well as the constant.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Factorise $2x^2 + 7x + 3$.

1.2 Factorise $3x^2 - 10x + 3$.

1.3 Factorise $4x^2 + 12x + 8$ fully.

1.4 Expand $(2x - 3)(3x + 4)$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Difference of squares and complete factorisation

The identity $a^2 - b^2 = (a - b)(a + b)$ follows because the middle terms cancel on expansion. Recognise squared expressions, not just square numbers: $9x^2 - 25$ is $(3x)^2 - 5^2$. A sum of squares does not factor by this identity over the real numbers.

Full factorisation can require several stages. Remove a common factor, apply a structural identity, then inspect the factors again. For example, $x^4 - 16$ contains a difference of squares, and one resulting factor contains another. Stop when the factors cannot be broken down further using the intended GCSE real or integer methods.

A reliable method

1. Look for a common factor before applying an identity.
2. Identify both squares explicitly.
3. Inspect the resulting factors for another possible step.

Worked example 1

Factorise $25x^2 - 49$.

STEP-BY-STEP SOLUTION

This is $(5x)^2 - 7^2$, so $(5x - 7)(5x + 7)$.

Worked example 2

Factorise $3x^3 - 12x$ fully.

STEP-BY-STEP SOLUTION

Take out $3x$: $3x(x^2 - 4)$. Then difference of squares gives $3x(x - 2)(x + 2)$.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

$x^2 + 9$ is not $(x + 3)(x - 3)$, because that product is $x^2 - 9$. The sign between the squares is essential.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Factorise $x^2 - 64$.

2.2 Factorise $16a^2 - b^2$.

2.3 Factorise $2x^3 - 50x$.

2.4 Factorise $x^4 - 81$ over the real numbers using GCSE methods.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Three brackets and algebraic structure

For three brackets, expand two first and collect like terms before multiplying by the third. Every term in the intermediate expression must multiply every term in the remaining bracket. Ordering powers in descending order helps prevent missing terms.

Sometimes a special pair makes the work shorter. Multiplying conjugate brackets such as $(x + 2)(x - 2)$ first removes the middle term. The constant term of a product is the product of the constants, and the leading term is the product of the leading terms. These observations give quick independent checks.

A reliable method

1. Choose the easiest pair of brackets first.

2. Multiply the collected result by every term in the final bracket.

3. Check the leading term, constant term and a simple input.

Worked example 1

Expand $(x + 1)(x + 2)(x + 3)$.

STEP-BY-STEP SOLUTION

First $(x + 1)(x + 2) = x^2 + 3x + 2$. Multiply by $x + 3$: $x^3 + 3x^2 + 3x^2 + 9x + 2x + 6 = x^3 + 6x^2 + 11x + 6$.

Worked example 2

Expand $(x - 2)(x + 2)(x - 5)$.

STEP-BY-STEP SOLUTION

First use difference of squares: $(x^2 - 4)(x - 5) = x^3 - 5x^2 - 4x + 20$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Substituting $x = 0$ checks only the constant term. It is useful but does not prove two expressions identical. Combine numerical checks with complete algebraic expansion.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Expand $(x + 1)(x - 1)(x + 4)$.

3.2 Expand $(x + 2)^3$.

3.3 Expand $(2x + 1)(x - 1)(x + 1)$.

3.4 Check the constant and leading terms of $(3x - 2)(x + 5)(x - 4)$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Factorise $6x^2 + 7x + 2$. [3 marks]

R2. Factorise $3x^2 - 8x + 4$. [3 marks]

R3. Factorise $49p^2 - 4q^2$. [2 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Factorise $5x^3 - 45x$ fully. [3 marks]

R5. Expand $(x - 1)(x + 2)(x + 5)$. [4 marks]

R6. Explain why $x^2 + 16$ cannot equal $(x - 4)(x + 4)$. [2 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Quadratics with a leading coefficient

1.1 Factorise $2x^2 + 7x + 3$.

Split $7x$ as $6x + x$: $(2x + 1)(x + 3)$.

1.2 Factorise $3x^2 - 10x + 3$.

Split $-10x$ as $-9x - x$: $(3x - 1)(x - 3)$.

1.3 Factorise $4x^2 + 12x + 8$ fully.

First take 4: $4(x^2 + 3x + 2) = 4(x + 1)(x + 2)$.

1.4 Expand $(2x - 3)(3x + 4)$.

$6x^2 + 8x - 9x - 12 = 6x^2 - x - 12$.

Lesson 2 • Difference of squares and complete factorisation

2.1 Factorise $x^2 - 64$.

$(x - 8)(x + 8)$.

2.2 Factorise $16a^2 - b^2$.

$(4a - b)(4a + b)$.

2.3 Factorise $2x^3 - 50x$.

$2x(x^2 - 25) = 2x(x - 5)(x + 5)$.

2.4 Factorise $x^4 - 81$ over the real numbers using GCSE methods.

$(x^2 - 9)(x^2 + 9) = (x - 3)(x + 3)(x^2 + 9)$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Three brackets and algebraic structure

3.1 Expand $(x + 1)(x - 1)(x + 4)$.

$(x^2 - 1)(x + 4) = x^3 + 4x^2 - x - 4$.

3.2 Expand $(x + 2)^3$.

$(x^2 + 4x + 4)(x + 2) = x^3 + 6x^2 + 12x + 8$.

3.3 Expand $(2x + 1)(x - 1)(x + 1)$.

$(2x + 1)(x^2 - 1) = 2x^3 + x^2 - 2x - 1$.

3.4 Check the constant and leading terms of $(3x - 2)(x + 5)(x - 4)$.

Leading term is $3x^3$. Constant is $(-2) \times 5 \times (-4) = 40$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Factorise $6x^2 + 7x + 2$. [3 marks]

Split using 3 and 4: $(3x + 2)(2x + 1)$. [1 for pair; 1 for grouping; 1 for factors.]

R2. Factorise $3x^2 - 8x + 4$. [3 marks]

Split $-8x$ as $-6x - 2x$: $(3x - 2)(x - 2)$. [1 for split; 1 for grouping; 1 for result.]

R3. Factorise $49p^2 - 4q^2$. [2 marks]

$(7p - 2q)(7p + 2q)$. [1 for recognising squares; 1 for factors.]

R4. Factorise $5x^3 - 45x$ fully. [3 marks]

$5x(x^2 - 9) = 5x(x - 3)(x + 3)$. [1 for common factor; 1 for identity; 1 for full result.]

R5. Expand $(x - 1)(x + 2)(x + 5)$. [4 marks]

$(x^2 + x - 2)(x + 5) = x^3 + 6x^2 + 3x - 10$. [1 for first product; 2 for distribution; 1 for collection.]

R6. Explain why $x^2 + 16$ cannot equal $(x - 4)(x + 4)$. [2 marks]

The product is $x^2 - 16$. At $x = 0$ the claimed equality would say $16 = -16$. [1 for expansion; 1 for explanation/check.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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ALGEBRA & GRAPHS / TOPIC GUIDE 15

15

Linear equations

Shared GCSE core

An equation is a balance: equivalent operations on both sides preserve its solutions. Progress from one-step equations to brackets, fractions and unknowns on both sides, then form equations from short problems.

$4x + 7 = 31 \rightarrow x = 6$

Balance both sides and verify by substitution.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Use inverse operations, expand brackets and calculate with signed numbers.

Your route through this notebook

1. Balancing and inverse operations
2. Brackets and unknowns on both sides
3. Fractions and forming equations

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Balancing and inverse operations

Solving means finding values that make an equation true. Adding or subtracting the same quantity on both sides preserves equality. Multiplying or dividing both sides by the same non-zero constant also preserves equality. These operations undo the operations on the unknown.

Work backwards through a calculation. In $3x + 5 = 20$, the final addition is undone before the multiplication. Show a new equation after every step instead of moving symbols without explanation. Check a solution by substituting it into the original equation, not only your final line.

A reliable method

1. Identify the operations applied to the unknown.

2. Undo them in reverse order on both sides.

3. Substitute the result into the original equation.

Worked example 1

Solve $4x + 7 = 31$.

STEP-BY-STEP SOLUTION

Subtract 7: $4x = 24$. Divide by 4: $x = 6$. Check: $4 \times 6 + 7 = 31$.

Worked example 2

Solve $x/5 - 3 = 4$.

STEP-BY-STEP SOLUTION

Add 3: $x/5 = 7$. Multiply by 5: $x = 35$.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

The equals sign connects expressions with the same value. Do not write $4x + 7 = 31 - 7 = 24$, because the first expression equals 31, not 24. Each balanced stage should be its own equation.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Solve $x + 9 = 17$.

1.2 Solve $6x = -42$.

1.3 Solve $5x - 8 = 22$.

1.4 Solve $x/4 + 2 = -1$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Brackets and unknowns on both sides

When x appears on both sides, collect all x terms on one side and constants on the other by balanced addition or subtraction. Either direction can work; choosing a positive final coefficient often simplifies the arithmetic. Expand brackets before collecting, taking care with negative multipliers.

An equation may have one solution, no solutions or infinitely many solutions. If all variable terms cancel and leave a false statement, no input works. If they leave an identity such as 5 = 5, every input in the original domain works. A cancelled x does not automatically mean x = 0.

A reliable method

1. Expand and simplify each side.

2. Subtract variable terms from both sides, then isolate the constant.

3. If variables disappear, interpret the remaining statement.

Worked example 1

Solve 5x + 4 = 2x + 19.

STEP-BY-STEP SOLUTION

Subtract 2x: 3x + 4 = 19. Subtract 4: 3x = 15. Therefore x = 5.

Worked example 2

Solve 3(2x − 1) = 4x + 9.

STEP-BY-STEP SOLUTION

Expand: 6x − 3 = 4x + 9. Then 2x = 12, so x = 6.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

For 2(x + 3) = 2x + 6, expansion leaves the same expression on both sides, so every real x works. For 2(x + 3) = 2x + 7, cancellation gives 6 = 7, so no real x works.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Solve $7x - 5 = 3x + 11$.

2.2 Solve $4(x + 2) = 3x + 13$.

2.3 Solve $5 - 2(x - 3) = x + 2$.

2.4 Solve $3x + 8 = 3x - 1$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Fractions and forming equations

For equations with fractional terms, multiplying every term on both sides by a common denominator can remove the fractions. The multiplier must apply to entire numerators. Brackets preserve this structure and reduce sign errors. When denominators are fixed non-zero numbers, this step preserves the solution set.

For a word problem, define the unknown, express each relevant quantity and state the relationship as an equation. After solving, check whether the answer fits the context: lengths must be positive and a count may need an integer. Writing an equation is part of the solution, not just a route to a number.

A reliable method

1. Use the LCM of fixed denominators to clear fractions.

2. Apply the multiplier to every term, preserving brackets.

3. For a model, define x and interpret its solution.

Worked example 1

Solve $(x + 2)/3 = (x - 1)/2$.

STEP-BY-STEP SOLUTION

Multiply by 6: $2(x + 2) = 3(x - 1)$. Thus $2x + 4 = 3x - 3$ and $x = 7$.

Worked example 2

A rectangle has width x cm and length $x + 4$ cm. Its perimeter is 36 cm. Find both sides.

STEP-BY-STEP SOLUTION

$2x + 2(x + 4) = 36$. Then $4x + 8 = 36$, so $x = 7$. Width 7 cm and length 11 cm.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Multiplying (x + 2)/3 by 6 gives 2(x + 2), not 2x + 2. The original fraction divides the entire numerator, so the inverse step must act on that entire group.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Solve $x/3 + 2 = 7$.

3.2 Solve $(2x - 1)/5 = 3$.

3.3 Solve $x/2 + x/3 = 10$.

3.4 Three consecutive integers sum to 72. Find them.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Solve $7x + 2 = 44$. [2 marks]

R2. Solve $9 - 3x = 24$. [2 marks]

R3. Solve $5(x - 2) = 2x + 8$. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Solve $(x + 1)/4 + x/2 = 4$. [3 marks]

R5. A number is doubled and increased by 9 to give 35. Find it. [3 marks]

R6. Solve $4(x + 2) = 4x + 8$ and explain. [2 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Balancing and inverse operations

1.1 Solve $x + 9 = 17$.

Subtract 9: $x = 8$.

1.2 Solve $6x = -42$.

Divide by 6: $x = -7$.

1.3 Solve $5x - 8 = 22$.

$5x = 30$, so $x = 6$.

1.4 Solve $x/4 + 2 = -1$.

$x/4 = -3$, so $x = -12$.

Lesson 2 • Brackets and unknowns on both sides

2.1 Solve $7x - 5 = 3x + 11$.

$4x = 16$, so $x = 4$.

2.2 Solve $4(x + 2) = 3x + 13$.

$4x + 8 = 3x + 13$, so $x = 5$.

2.3 Solve $5 - 2(x - 3) = x + 2$.

$11 - 2x = x + 2$; $9 = 3x$, so $x = 3$.

2.4 Solve $3x + 8 = 3x - 1$.

Subtract $3x$: $8 = -1$ is false, so there is no solution.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Fractions and forming equations

3.1 Solve x/3 + 2 = 7.

x/3 = 5, so x = 15.

3.2 Solve (2x − 1)/5 = 3.

2x − 1 = 15; x = 8.

3.3 Solve x/2 + x/3 = 10.

Multiply by 6: 3x + 2x = 60. Thus x = 12.

3.4 Three consecutive integers sum to 72. Find them.

Let the first be n. n + (n + 1) + (n + 2) = 72 gives 3n = 69, so 23, 24, 25.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Solve $7x + 2 = 44$. [2 marks]

$7x = 42$, so $x = 6$. [1 for subtracting 2; 1 for division.]

R2. Solve $9 - 3x = 24$. [2 marks]

$-3x = 15$, so $x = -5$. [1 for isolation; 1 for result.]

R3. Solve $5(x - 2) = 2x + 8$. [3 marks]

$5x - 10 = 2x + 8$; $3x = 18$; $x = 6$. [1 for expansion; 1 for collection; 1 for result.]

R4. Solve $(x + 1)/4 + x/2 = 4$. [3 marks]

Multiply by 4: $x + 1 + 2x = 16$. Then $3x = 15$ and $x = 5$. [1 for clearing fractions; 1 for collection; 1 for result.]

R5. A number is doubled and increased by 9 to give 35. Find it. [3 marks]

$2n + 9 = 35$; $2n = 26$; $n = 13$. [1 for model; 1 for balancing; 1 for result.]

R6. Solve $4(x + 2) = 4x + 8$ and explain. [2 marks]

Expansion gives $4x + 8 = 4x + 8$, true for every real x. [1 for expansion; 1 for all-real conclusion.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 16

16

Formulae and changing the subject

Shared core with Higher extensions

A formula describes a relationship between quantities. Learn to evaluate it, build it from a context and rearrange it without losing its meaning. Higher extensions include powers and a subject appearing more than once.

$$y = 3x + 7 \rightarrow x = (y - 7)/3$$

Undo operations while preserving the relationship.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Substitute values, solve linear equations and factorise common terms.

Your route through this notebook

1. Using and constructing formulae
2. Rearranging with inverse operations
3. Powers, roots and repeated subjects — Higher

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Using and constructing formulae

A formula usually connects several quantities that can vary. Identify the subject, typically the isolated variable on the left, and the units of every input. Substitution must use compatible units: a speed in metres per second requires distance in metres and time in seconds.

To construct a formula, separate constants from variable relationships. Repeated equal groups suggest multiplication; a shared total may suggest division. State what each letter represents. A formula is not automatically valid for every real input: denominators cannot be zero and geometric dimensions normally must be positive.

A reliable method

1. Define letters and use consistent units.

2. Translate the relationship before substituting numbers.

3. Interpret the output in the required unit.

Worked example 1

Use v = d/t for d = 240 m and t = 30 s.

STEP-BY-STEP SOLUTION

v = 240/30 = 8 m/s. Both inputs are already in compatible units.

Worked example 2

A rental costs a fixed £18 and £6 per day. Write a formula and find the cost for 4 days.

STEP-BY-STEP SOLUTION

Let d be days and C be cost in pounds. C = 18 + 6d; C = 42 for d = 4.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

The letters used in a formula are choices, not magic labels. A different author may use T instead of C for a cost. Definitions and units establish the meaning.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Use A = bh/2 for b = 12 cm and h = 7 cm.

1.2 Use v = d/t for d = 1.2 km and t = 4 minutes, giving m/s.

1.3 A tank holds 150 litres and fills at r litres per minute from empty. Express time t.

1.4 A rectangle has width w and length 2w + 3. Express its perimeter.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Rearranging with inverse operations

Changing the subject means isolating a different variable while retaining an equivalent relationship. Treat the other letters as quantities and apply the same balanced operations used in equations. Avoid moving symbols without naming the operation: division is not simply “moving a letter underneath”.

Grouped expressions must stay grouped. In $y = (x + a)/b$, multiply by b before subtracting a. If a variable is in a denominator, multiply first to remove that denominator, respecting its non-zero restriction. Rearranging does not remove any domain conditions of the original formula.

A reliable method

1. Identify the new subject and the operations surrounding it.

2. Undo operations on both sides in reverse order.

3. Use brackets for whole numerators and denominators.

Worked example 1

Make x the subject of $y = 3x + 7$.

STEP-BY-STEP SOLUTION

Subtract 7: $y - 7 = 3x$. Divide by 3: $x = (y - 7)/3$.

Worked example 2

Make t the subject of $v = d/t$.

STEP-BY-STEP SOLUTION

Multiply by t: $vt = d$. Divide by non-zero v: $t = d/v$. The original also requires $t \neq 0$.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

From $y = 5x + 2$, $x = y/5 - 2$ is generally wrong because only y has been divided. The correct expression is $(y - 2)/5$. Substituting a simple original pair checks the rearrangement.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Make x the subject of $y = 4x - 9$.

2.2 Make h the subject of $A = bh/2$.

2.3 Make a the subject of $v = u + at$.

2.4 Make x the subject of $y = (x - 3)/5$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Powers, roots and repeated subjects — Higher

When the subject is squared, isolate its square before taking roots. Algebraically both signs may be needed; in a length formula the context often selects the positive root. For example, $x^2 = k$ gives $x = \pm\sqrt{k}$ for $k \geq 0$, but a radius is non-negative.

If the desired subject occurs in several terms, collect those terms and factorise. The subject can then be isolated by dividing by the remaining bracket. If that bracket could equal zero, the rearranged form has a restriction and the original relationship needs separate consideration in that case.

A reliable method

1. Isolate powers before taking roots and consider signs.

2. Collect all terms containing the subject on one side.

3. Factorise the subject, divide and state non-zero restrictions.

Worked example 1

Make r the subject of $A = \pi r^2$ for a circle.

STEP-BY-STEP SOLUTION

$r^2 = A/\pi$. Since r is a radius, $r = \sqrt{(A/\pi)}$, with $A \geq 0$.

Worked example 2

Make x the subject of $y = ax + bx + c$.

STEP-BY-STEP SOLUTION

$y - c = x(a + b)$. Thus $x = (y - c)/(a + b)$, provided $a + b \neq 0$. If $a + b = 0$, the original reduces to $y = c$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Taking a square root of a sum does not split it into separate roots. From $x^2 = p + q$, $x = \pm\sqrt{(p + q)}$, not $\pm(\sqrt{p} + \sqrt{q})$.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Make x the subject of $y = 2x^2 + 3$ over real numbers.

3.2 Make x the subject of $p = 5x - qx$.

3.3 Make x the subject of $y = (x + 2)/(x - 1)$.

3.4 Make r the subject of $V = \pi r^2 h$ for a cylinder with $h > 0$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Use $C = 5n + 12$ for $n = 9$. [2 marks]

R2. Make x the subject of $y = 7x + 4$. [2 marks]

R3. Make b the subject of $P = 2a + 2b$. [2 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Higher: make x the subject of $y = 4x^2 - 5$. [3 marks]

R5. Higher: make x the subject of $p = ax - b(x + 1)$. [3 marks]

R6. Higher: make x the subject of $y = (2x + 3)/(x - 4)$. [4 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Using and constructing formulae

1.1 Use A = bh/2 for b = 12 cm and h = 7 cm.

A = 12 × 7 / 2 = 42 cm².

1.2 Use v = d/t for d = 1.2 km and t = 4 minutes, giving m/s.

1.2 km = 1200 m; 4 min = 240 s; v = 5 m/s.

1.3 A tank holds 150 litres and fills at r litres per minute from empty. Express time t.

t = 150/r minutes, for r > 0.

1.4 A rectangle has width w and length 2w + 3. Express its perimeter.

P = 2w + 2(2w + 3) = 6w + 6.

Lesson 2 • Rearranging with inverse operations

2.1 Make x the subject of y = 4x − 9.

x = (y + 9)/4.

2.2 Make h the subject of A = bh/2.

2A = bh, so h = 2A/b for b ≠ 0.

2.3 Make a the subject of v = u + at.

a = (v − u)/t, for t ≠ 0.

2.4 Make x the subject of y = (x − 3)/5.

5y = x − 3, so x = 5y + 3.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Powers, roots and repeated subjects — Higher

3.1 Make x the subject of $y = 2x^2 + 3$ over real numbers.

$x^2 = (y - 3)/2$; $x = \pm\sqrt{((y - 3)/2)}$, requiring $y \geq 3$.

3.2 Make x the subject of $p = 5x - qx$.

$p = x(5 - q)$, so $x = p/(5 - q)$, if $q \neq 5$.

3.3 Make x the subject of $y = (x + 2)/(x - 1)$.

$y(x - 1) = x + 2$; $x(y - 1) = y + 2$; $x = (y + 2)/(y - 1)$, with $y \neq 1$ and original $x \neq 1$.

3.4 Make r the subject of $V = \pi r^2 h$ for a cylinder with $h > 0$.

$r = \sqrt{(V/(\pi h))}$, taking the non-negative radius for $V \geq 0$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Use $C = 5n + 12$ for $n = 9$. [2 marks]

$C = 45 + 12 = 57$. [1 for substitution; 1 for result.]

R2. Make x the subject of $y = 7x + 4$. [2 marks]

$x = (y - 4)/7$. [1 for subtraction; 1 for division.]

R3. Make b the subject of $P = 2a + 2b$. [2 marks]

$b = (P - 2a)/2$, or $P/2 - a$. [1 for isolation; 1 for division.]

R4. Higher: make x the subject of $y = 4x^2 - 5$. [3 marks]

$x = \pm\sqrt{((y + 5)/4)}$, for $y \geq -5$. [1 for isolating square; 1 for root; 1 for both signs/domain.]

R5. Higher: make x the subject of $p = ax - b(x + 1)$. [3 marks]

$p = (a - b)x - b$; $x = (p + b)/(a - b)$, for $a \neq b$. [1 for expansion; 1 for collecting; 1 for division/restriction.]

R6. Higher: make x the subject of $y = (2x + 3)/(x - 4)$. [4 marks]

$yx - 4y = 2x + 3$; $x(y - 2) = 4y + 3$; $x = (4y + 3)/(y - 2)$, with $y \neq 2$ and original $x \neq 4$. [1 for each algebraic stage; 1 for restrictions.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 17

17

Inequalities

Shared core with Higher extensions

Inequalities describe sets of possible values rather than a single answer. Learn number-line notation and balanced solving, then extend to shaded coordinate regions and quadratic inequalities.

$$-2x > 8 \rightarrow x < -4$$

Dividing by a negative reverses the inequality.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

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Before you begin

Solve linear equations, use signed arithmetic and plot straight lines.

Your route through this notebook

1. Number lines and solving linear inequalities
2. Combined inequalities and graphical regions — extension
3. Quadratic inequalities — Higher

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Number lines and solving linear inequalities

The symbols < and > exclude equality; ≤ and ≥ include it. On a number line an open circle excludes an endpoint and a filled circle includes it. An arrow or line shows all allowed values, not only integers. If integer solutions are requested, select the integers within the solution interval.

Adding or subtracting the same quantity preserves an inequality. Multiplying or dividing by a negative number reverses its direction: $2 < 5$ but $-2 > -5$. This reversal follows from reversing the order on the number line. Division by zero is not permitted.

A reliable method

1. Solve using balanced operations.

2. Reverse the inequality when multiplying or dividing by a negative.

3. Represent the entire solution set, checking endpoint inclusion.

Worked example 1

Solve $3x - 4 \leq 11$.

STEP-BY-STEP SOLUTION

$3x \leq 15$, so $x \leq 5$. A filled circle at 5 and shading left represent all solutions.

Worked example 2

Solve $-2x + 7 > 15$.

STEP-BY-STEP SOLUTION

$-2x > 8$. Dividing by -2 reverses the sign: $x < -4$.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

$x < 4$ does not mean $x = 3$. It includes 3.9, zero and every real value below 4. A list of integers is a narrower answer and needs an integer condition.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Solve $x + 6 > 10$.

1.2 Solve $5x \leq 20$.

1.3 Solve $-3x \geq 12$.

1.4 List integer values satisfying $-2 < x \leq 3$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Combined inequalities and graphical regions — extension

A double inequality can constrain a value from both sides. Apply additions, subtractions or positive divisions to all three parts. For more complicated combinations, solve each inequality separately and take their intersection: values must satisfy both.

In two variables, an inequality selects a region of the coordinate plane. Draw the boundary equation first. A strict inequality has a dashed boundary; an inclusive inequality has a solid boundary. Test a point not on the line to decide which side satisfies the condition. A system of inequalities uses the overlap of all valid regions.

A reliable method

1. Solve both parts of a combined condition.

2. Draw each boundary with the correct solid or dashed style.

3. Test a point, then retain the overlap.

Worked example 1

Solve $2 < 3x + 5 \leq 14$.

STEP-BY-STEP SOLUTION

Subtract 5 throughout: $-3 < 3x \leq 9$. Divide by 3: $-1 < x \leq 3$.

Worked example 2

Describe $y \geq 2x + 1$.

STEP-BY-STEP SOLUTION

The boundary $y = 2x + 1$ is solid. At (0,0), $0 \geq 1$ is false. Keep the region above the line, including the line.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

“Shade the region” can mean either shade the wanted region or shade out unwanted regions depending on the instructions. State your convention and label the final valid region clearly.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Solve $-5 \leq 2x + 1 < 7$.

2.2 Solve $x > 1$ and $x \leq 6$.

2.3 Describe the boundary and region for $x < 3$.

2.4 Does (2,5) satisfy $y \geq x + 2$ and $x \geq 0$?

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Quadratic inequalities — Higher

A quadratic inequality asks where a quadratic expression is positive or negative. Find its roots and examine the sign between and outside them. For a positive leading coefficient, a parabola opens upwards: it is below the axis between two distinct roots and above it outside. A negative leading coefficient reverses that pattern.

A sketch or test values make the reasoning visible. Include roots for ≤ or ≥, and exclude them for < or >. The roots alone are boundary values, not the entire solution set. Repeated roots and quadratics with no real roots need a separate sign check.

A reliable method

1. Rearrange to compare a quadratic with zero.

2. Find roots and determine the opening direction.

3. Select intervals of the required sign, with correct endpoint inclusion.

Worked example 1

Solve $x^2 - 5x + 6 < 0$.

STEP-BY-STEP SOLUTION

Factorise $(x - 2)(x - 3)$. Roots are 2 and 3; the upward parabola is negative between them. Thus $2 < x < 3$.

Worked example 2

Solve $x^2 - 9 \geq 0$.

STEP-BY-STEP SOLUTION

Roots are -3 and 3. The expression is non-negative outside the interval, including endpoints: $x \leq -3$ or $x \geq 3$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Do not write $x < 2$ and $x > 3$ for an outside region: no number satisfies both. Outside intervals usually require “or”; an inside interval requires “and”.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Solve $(x - 1)(x - 4) \leq 0$.

3.2 Solve $x^2 - 16 > 0$.

3.3 Solve $x^2 + 1 > 0$ over real numbers.

3.4 Solve $-x^2 + 4 \geq 0$.

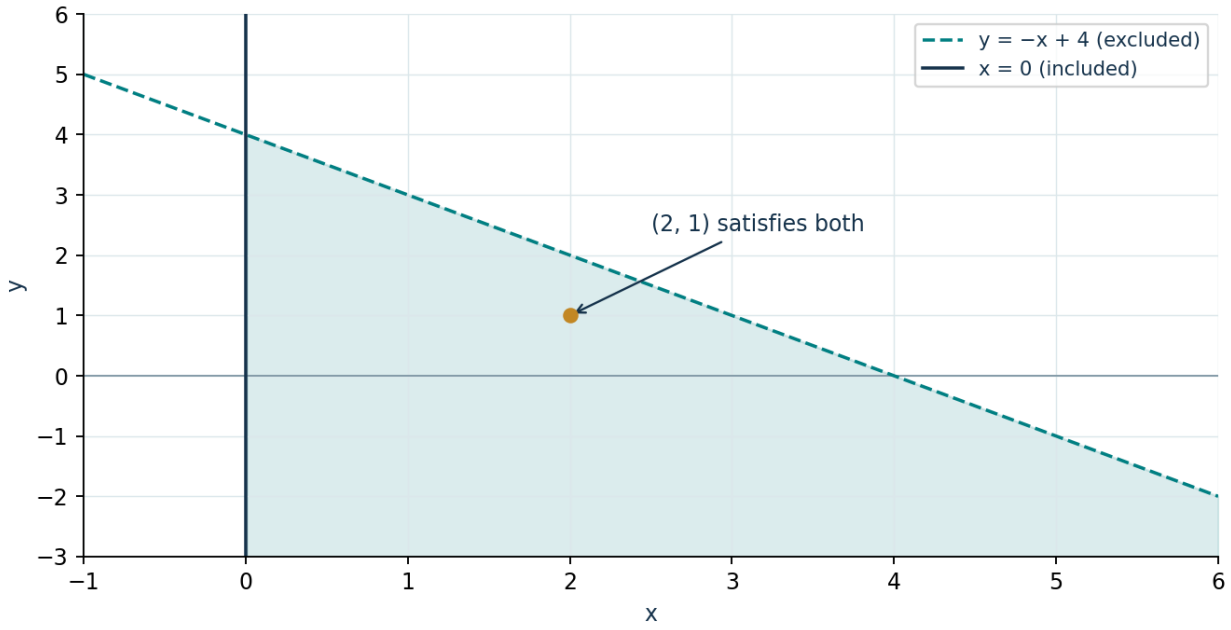
Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

An inequality selects a whole region



WHAT THE FIGURE SHOWS

The shaded overlap satisfies $y < -x + 4$ and $x \geq 0$. The dashed sloping boundary is excluded; the vertical boundary is included wherever the other inequality holds. The point (2,1) is an example, not the whole solution set.

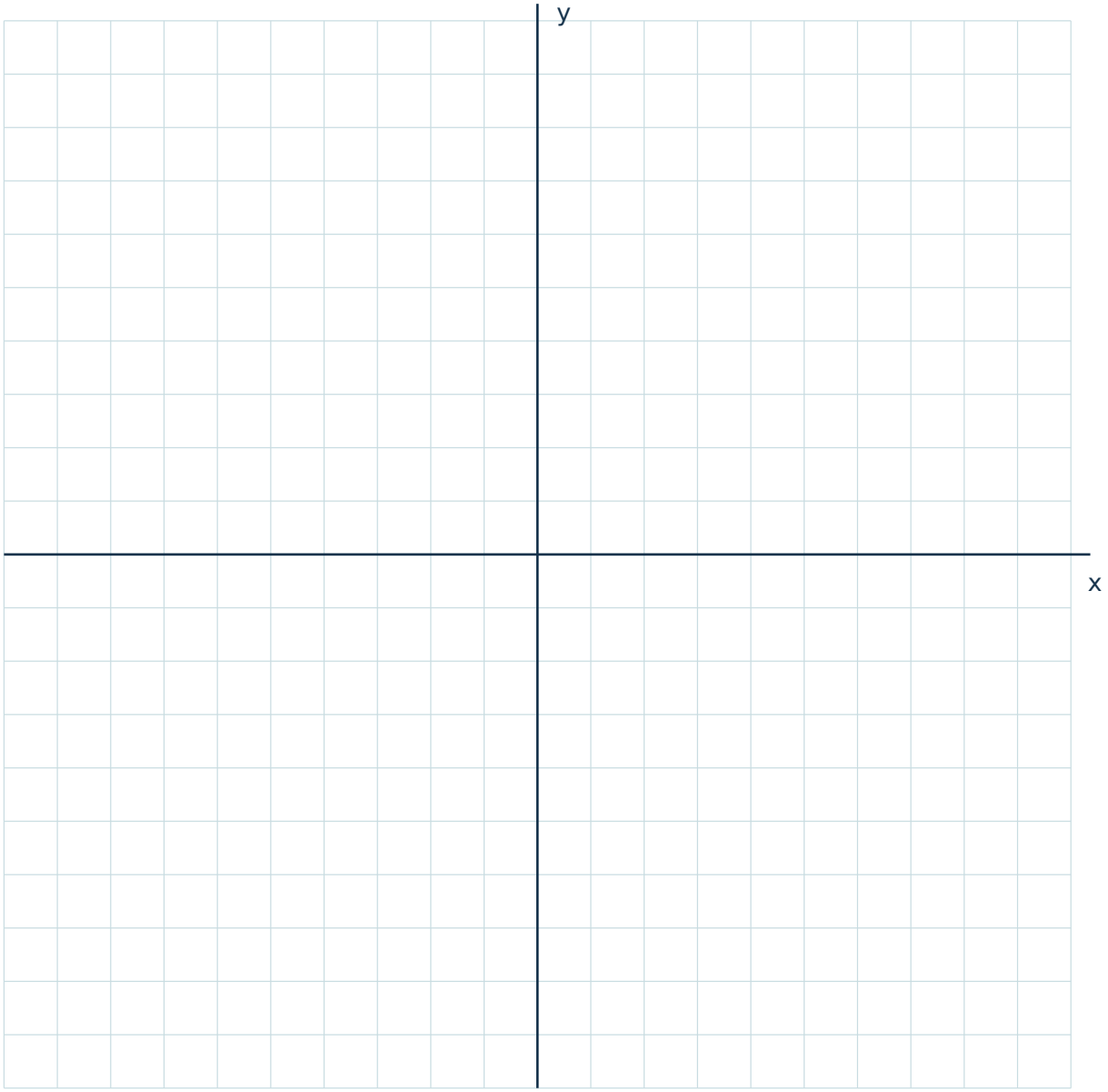
Explain how the labelled features follow from the equations. Check a point by substitution or recalculate a gradient or area before relying on the picture.

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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

Choose one graph from the preceding visual explainer and reconstruct it on this grid. Choose and label a suitable scale on each axis; one square need not represent one unit. Include the displayed domain, key points and units. For a real-life graph, relabel x and y with the measured quantities.



SELF-CHECK

Compare with the preceding plotted figure and its explanation. Check at least three points, any intercepts and the direction or shape of the graph. Explain how the equation or context produces those features.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Solve $4x - 3 < 17$. [2 marks]

R2. Solve $8 - 2x \leq 14$. [2 marks]

R3. List integer solutions of $-1 \leq x < 4$. [1 mark]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Solve $1 < 2x + 3 \leq 11$. [3 marks]

R5. Higher: describe the region $y < -x + 4$, $x \geq 0$. [3 marks]

R6. Higher: solve $x^2 - x - 6 \geq 0$. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Number lines and solving linear inequalities

1.1 Solve $x + 6 > 10$.

$x > 4$.

1.2 Solve $5x \leq 20$.

$x \leq 4$.

1.3 Solve $-3x \geq 12$.

Divide by -3 and reverse: $x \leq -4$.

1.4 List integer values satisfying $-2 < x \leq 3$.

$-1, 0, 1, 2, 3$.

Lesson 2 • Combined inequalities and graphical regions — extension

2.1 Solve $-5 \leq 2x + 1 < 7$.

Subtract 1 and divide by 2: $-3 \leq x < 3$.

2.2 Solve $x > 1$ and $x \leq 6$.

$1 < x \leq 6$.

2.3 Describe the boundary and region for $x < 3$.

A dashed vertical line at $x = 3$; the valid region is to its left.

2.4 Does (2,5) satisfy $y \geq x + 2$ and $x \geq 0$?

Yes: $5 \geq 4$ and $2 \geq 0$. Both conditions hold.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Quadratic inequalities — Higher

3.1 Solve $(x - 1)(x - 4) \leq 0$.

$1 \leq x \leq 4$.

3.2 Solve $x^2 - 16 > 0$.

$x < -4$ or $x > 4$.

3.3 Solve $x^2 + 1 > 0$ over real numbers.

All real x, because $x^2 \geq 0$ so $x^2 + 1$ is always positive.

3.4 Solve $-x^2 + 4 \geq 0$.

$x^2 \leq 4$, so $-2 \leq x \leq 2$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Solve $4x - 3 < 17$. [2 marks]

$x < 5$. [1 for isolation; 1 for division.]

R2. Solve $8 - 2x \leq 14$. [2 marks]

$-2x \leq 6$, so $x \geq -3$. [1 for isolation; 1 for reversal/result.]

R3. List integer solutions of $-1 \leq x < 4$. [1 mark]

$-1, 0, 1, 2, 3$. [1 for complete list.]

R4. Solve $1 < 2x + 3 \leq 11$. [3 marks]

$-2 < 2x \leq 8$, so $-1 < x \leq 4$. [1 for each balanced stage; 1 for endpoints.]

R5. Higher: describe the region $y < -x + 4$, $x \geq 0$. [3 marks]

Below the dashed line $y = -x + 4$ and on or to the right of the solid y-axis. [1 for each boundary; 1 for overlap.]

R6. Higher: solve $x^2 - x - 6 \geq 0$. [3 marks]

$(x - 3)(x + 2) \geq 0$, so $x \leq -2$ or $x \geq 3$. [1 for factors; 1 for sign regions; 1 for inclusivity.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 18

18

Simultaneous equations

Shared core with Higher extensions

Two equations can constrain the same pair of unknowns. Find values that satisfy both using elimination, substitution and graph intersections. Higher work combines a line with a quadratic.

$y = x^2$ and $y = x + 2$

Intersections satisfy both equations at once.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

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Before you begin

Solve linear equations, expand brackets and factorise simple quadratics.

Your route through this notebook

1. Elimination with linear equations
2. Substitution, models and graphical meaning
3. Linear–quadratic systems — Higher

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Elimination with linear equations

A solution of simultaneous equations must make both equations true at once. Elimination creates a new equation with only one unknown by adding or subtracting suitable multiples of the original equations. This is valid because equal quantities can be added to equal quantities.

First choose which variable to eliminate. Matching coefficients are cancelled by subtraction; opposite coefficients cancel by addition. If scaling is needed, multiply every term on that equation, including the right side. After finding one variable, substitute into an original equation and check the pair in both originals.

A reliable method

1. Align x terms, y terms and constants.

2. Scale to equal or opposite coefficients, then eliminate.

3. Back-substitute and check both equations.

Worked example 1

Solve $x + y = 9$ and $x - y = 3$.

STEP-BY-STEP SOLUTION

Add equations: $2x = 12$, so $x = 6$. Then $y = 3$. Both equations check.

Worked example 2

Solve $2x + 3y = 17$ and $3x + 2y = 18$.

STEP-BY-STEP SOLUTION

Multiply the first by 3 and second by 2: $6x + 9y = 51$; $6x + 4y = 36$. Subtract: $5y = 15$, so $y = 3$. Then $2x + 9 = 17$ gives $x = 4$.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Subtract an entire equation, not just its first term. Writing brackets around the row being subtracted helps protect negative signs.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Solve $x + y = 11$ and $x - y = 5$.

1.2 Solve $2x + y = 10$ and $x + y = 7$.

1.3 Solve $3x + 2y = 16$ and $x + 2y = 8$.

1.4 Solve $2x + 5y = 9$ and $3x - 5y = 11$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Substitution, models and graphical meaning

Substitution replaces one unknown by an equivalent expression from the other equation. It is especially efficient when an equation already has x or y isolated. Substitute the whole expression with brackets, solve the resulting one-variable equation, then recover the second unknown.

Each linear equation represents a line. An intersection supplies a pair satisfying both. Parallel distinct lines give no solution; the same line written in two ways gives infinitely many solutions. In modelling, define the quantities and use two independent relationships, such as a total count and a total cost.

A reliable method

1. Isolate a convenient variable.

2. Substitute the entire expression into the other equation.

3. Interpret the pair as a common solution or intersection.

Worked example 1

Solve $y = 2x + 1$ and $x + y = 10$.

STEP-BY-STEP SOLUTION

$x + (2x + 1) = 10$ gives $3x = 9$ and $x = 3$. Then $y = 7$.

Worked example 2

Three adult tickets and two child tickets cost £34. One adult and two children cost £18. Find each price.

STEP-BY-STEP SOLUTION

Let adult price be a and child price c. Subtract $a + 2c = 18$ from $3a + 2c = 34$: $2a = 16$, so $a = 8$. Then $2c = 10$, so $c = 5$.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A graph gives approximate readings unless intersections fall exactly on readable grid points. An algebraic solution can provide exact values. Always distinguish a coordinate pair (x,y) from two unrelated answers.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Solve $y = x + 4$ and $2x + y = 13$.

2.2 Solve $x = 3y$ and $x + y = 20$.

2.3 How many solutions have $y = 2x + 1$ and $y = 2x - 3$?

2.4 How many solutions have $x + y = 6$ and $2x + 2y = 12$?

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Linear–quadratic systems — Higher

A line and a quadratic can intersect twice, once or not at all over the real numbers. Substitute the linear expression into the quadratic relationship to obtain a quadratic equation. Solve it and find a matching second coordinate for each first coordinate.

Keep pairs together. A list of two x values and two y values without pairing can create false combinations. If a quadratic has no real roots, the graphs have no real intersection. A repeated root corresponds to tangency in the usual line–parabola situation.

A reliable method

1. Use the line to express one variable.

2. Substitute and solve the resulting quadratic.

3. Find and check a complete pair for each root.

Worked example 1

Solve $y = x + 2$ and $y = x^2$.

STEP-BY-STEP SOLUTION

Set $x^2 = x + 2$: $(x - 2)(x + 1) = 0$. For $x = 2$, $y = 4$; for $x = -1$, $y = 1$. Solutions: $(2,4)$, $(-1,1)$.

Worked example 2

Solve $x + y = 5$ and $xy = 6$.

STEP-BY-STEP SOLUTION

$y = 5 - x$, so $x(5 - x) = 6$. Rearrange $x^2 - 5x + 6 = 0$, giving $x = 2$ or 3. Pairs are $(2,3)$, $(3,2)$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

An intersection question is asking where both graph rules hold, not where either graph crosses an axis. Equate the y expressions or substitute one relation into the other.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Solve $y = x + 6$ and $y = x^2$.

3.2 Solve $y = 2x$ and $y = x^2$.

3.3 Solve $y = 2x - 1$ and $y = x^2$.

3.4 Solve $y = -1$ and $y = x^2$ over real numbers.

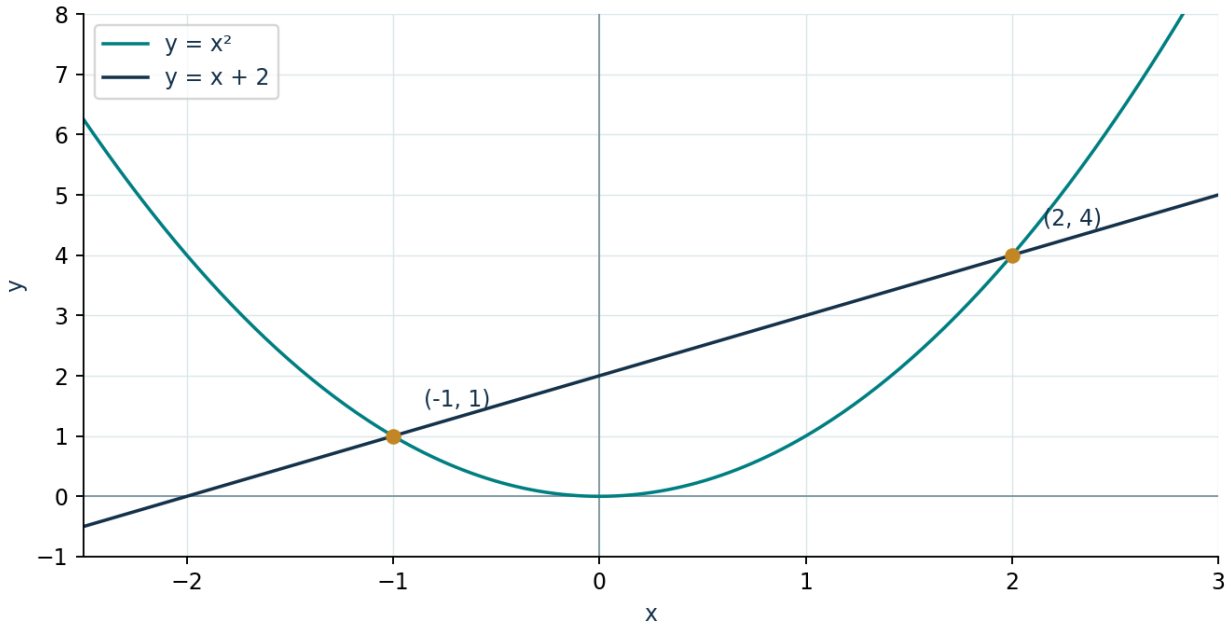
Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

Intersections solve both equations



WHAT THE FIGURE SHOWS

The two intersections are (−1,1) and (2,4). Each pair satisfies $y = x^2$ and $y = x + 2$. Algebra confirms them by solving $x^2 - x - 2 = 0$. The graph does not pair $x = -1$ with $y = 4$.

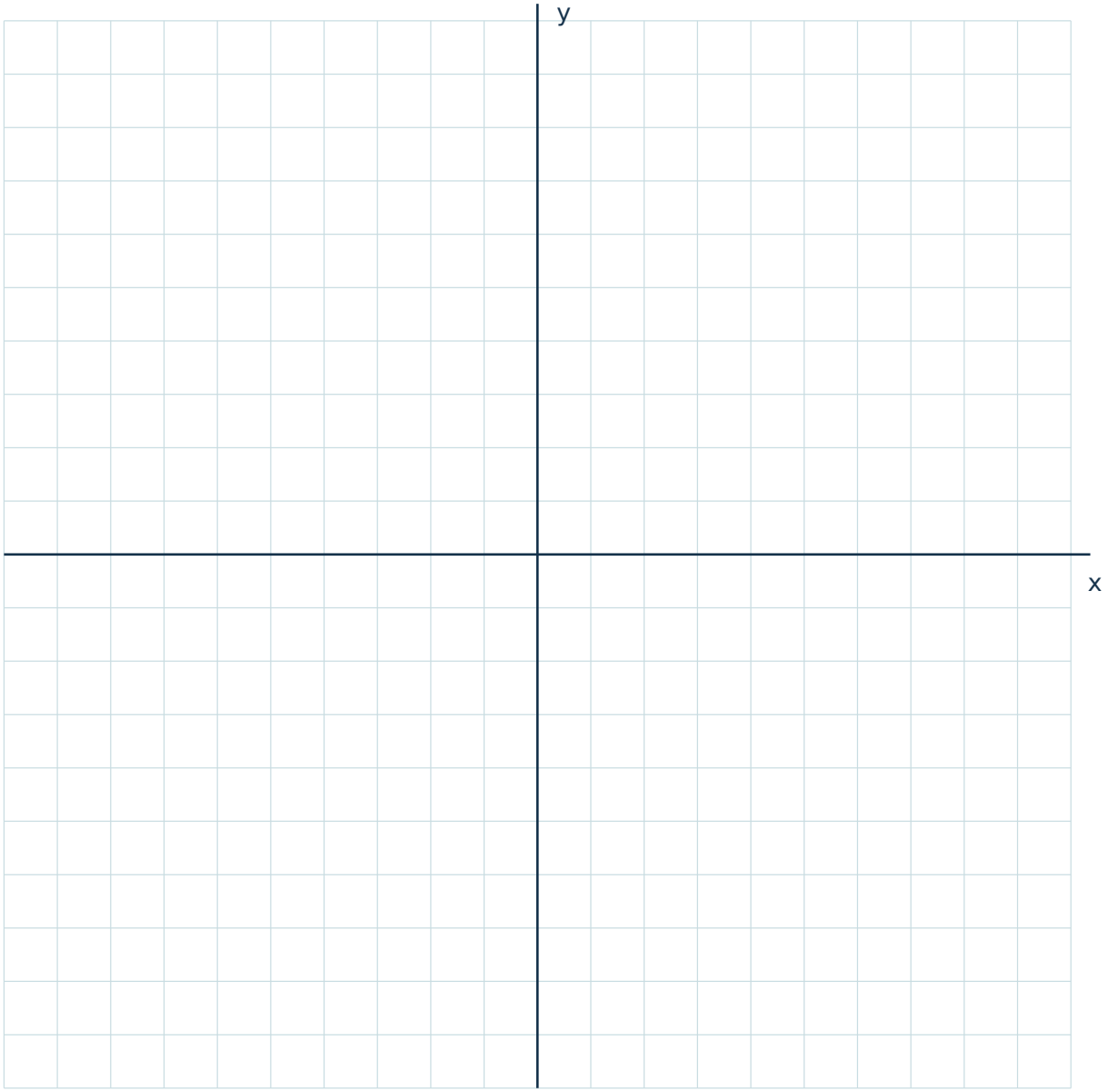
Explain how the labelled features follow from the equations. Check a point by substitution or recalculate a gradient or area before relying on the picture.

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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

Choose one graph from the preceding visual explainer and reconstruct it on this grid. Choose and label a suitable scale on each axis; one square need not represent one unit. Include the displayed domain, key points and units. For a real-life graph, relabel x and y with the measured quantities.



SELF-CHECK

Compare with the preceding plotted figure and its explanation. Check at least three points, any intercepts and the direction or shape of the graph. Explain how the equation or context produces those features.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Solve $x + y = 14$ and $x - y = 4$. [3 marks]

R2. Solve $3x + 2y = 19$ and $x + 2y = 9$. [3 marks]

R3. Solve $y = 3x - 2$ and $x + y = 14$. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Two pens and one book cost £9. One pen and one book cost £7. Find each price. [3 marks]

R5. Higher: solve $y = x + 12$ and $y = x^2$. [4 marks]

R6. Higher: solve $x + y = 7$ and $xy = 12$. [4 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Elimination with linear equations

1.1 Solve $x + y = 11$ and $x - y = 5$.

Add: $2x = 16$, so $x = 8$; $y = 3$.

1.2 Solve $2x + y = 10$ and $x + y = 7$.

Subtract: $x = 3$; then $y = 4$.

1.3 Solve $3x + 2y = 16$ and $x + 2y = 8$.

Subtract: $2x = 8$, so $x = 4$; $y = 2$.

1.4 Solve $2x + 5y = 9$ and $3x - 5y = 11$.

Add: $5x = 20$, so $x = 4$; then $8 + 5y = 9$, so $y = 1/5$.

Lesson 2 • Substitution, models and graphical meaning

2.1 Solve $y = x + 4$ and $2x + y = 13$.

$3x + 4 = 13$, so $x = 3$ and $y = 7$.

2.2 Solve $x = 3y$ and $x + y = 20$.

$4y = 20$, so $y = 5$ and $x = 15$.

2.3 How many solutions have $y = 2x + 1$ and $y = 2x - 3$?

None: equal gradients and different intercepts give parallel distinct lines.

2.4 How many solutions have $x + y = 6$ and $2x + 2y = 12$?

Infinitely many: the second equation is twice the first and represents the same line.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Linear–quadratic systems — Higher

3.1 Solve $y = x + 6$ and $y = x^2$.

$x^2 - x - 6 = 0$ gives $x = 3$ or -2. Pairs: (3,9), (−2,4).

3.2 Solve $y = 2x$ and $y = x^2$.

$x^2 = 2x$ gives $x(x - 2) = 0$. Pairs: (0,0), (2,4).

3.3 Solve $y = 2x - 1$ and $y = x^2$.

$x^2 - 2x + 1 = 0$, so $x = 1$ and $y = 1$. One solution: (1,1).

3.4 Solve $y = -1$ and $y = x^2$ over real numbers.

No real solution, because x^2 cannot equal -1.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Solve $x + y = 14$ and $x - y = 4$. [3 marks]

$2x = 18$, so $x = 9$ and $y = 5$. [1 for elimination; 1 for each value.]

R2. Solve $3x + 2y = 19$ and $x + 2y = 9$. [3 marks]

$2x = 10$; $x = 5$ and $y = 2$. [1 for elimination; 1 for each value.]

R3. Solve $y = 3x - 2$ and $x + y = 14$. [3 marks]

$4x - 2 = 14$, so $x = 4$ and $y = 10$. [1 for substitution; 1 for each value.]

R4. Two pens and one book cost £9. One pen and one book cost £7. Find each price. [3 marks]

$2p + b = 9$ and $p + b = 7$. Subtract: $p = 2$; then $b = 5$. [1 for model; 1 for each price.]

R5. Higher: solve $y = x + 12$ and $y = x^2$. [4 marks]

$x^2 - x - 12 = 0$ gives $(x - 4)(x + 3) = 0$. Pairs: $(4,16)$, $(-3,9)$. [1 for quadratic; 1 for roots; 1 for each pair.]

R6. Higher: solve $x + y = 7$ and $xy = 12$. [4 marks]

$x(7 - x) = 12$ leads to $x^2 - 7x + 12 = 0$. Pairs: $(3,4)$, $(4,3)$. [1 for substitution; 1 for quadratic; 1 for roots; 1 for pairing.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 19

19

Sequences

Shared GCSE core

A sequence is an ordered list generated by a rule. Learn to distinguish term-to-term rules from position-to-term rules, find linear nth terms and test membership. Include familiar number patterns without assuming every pattern is linear.

5, 8, 11, 14, … → 3n + 2

A linear nth term has a constant first difference.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Use signed arithmetic, multiplication and simple linear equations.

Your route through this notebook

1. Generating terms and recognising patterns
2. Finding a linear nth term
3. Membership, missing terms and number patterns

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Generating terms and recognising patterns

A term-to-term rule describes how to get the next term from earlier terms. It needs starting information: “add 4” alone does not fix a sequence. A position-to-term rule calculates a term directly from its position n, usually starting at n = 1. These two kinds of rule answer different questions.

Familiar patterns include square, cube and triangular numbers. Fibonacci-type sequences add the previous two terms, so they need two starting values. A finite list can fit many possible rules; use the stated context or rule rather than claiming a unique continuation from a few terms alone.

A reliable method

1. Identify the starting value or values.

2. Apply the given rule consistently.

3. Distinguish the position n from the value of the nth term.

Worked example 1

Start at 7 and subtract 3 each time. Write five terms.

STEP-BY-STEP SOLUTION

7, 4, 1, −2, −5.

Worked example 2

The nth term is $n^2 + 1$. Find the first four terms.

STEP-BY-STEP SOLUTION

Substitute n = 1,2,3,4: 2, 5, 10, 17.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

The sequence 2,4,6,8 has a natural “add 2” continuation, but four terms alone do not prove it. In an exam-style task, use the intended simple pattern and state your rule.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Start at −2 and add 5. Write five terms.

1.2 Find the first four terms of $3n - 1$.

1.3 A Fibonacci-type sequence starts 2,5. Write the next four terms.

1.4 Write the first five positive square numbers.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Finding a linear nth term

A linear sequence has a constant first difference d. Its nth term has form dn + c. The coefficient d is the difference; the constant c shifts the multiples of d to the required sequence. Compare the first term with d × 1 to find c.

Negative and fractional differences work in the same way. An nth-term formula allows direct access to a distant term without generating every earlier term. Check at least the first and another listed position. The formula describes positive integer positions unless another domain is stated.

A reliable method

1. Find the constant difference d.

2. Compare the sequence with d, 2d, 3d, … to find c.

3. Check the rule using known positions.

Worked example 1

Find the nth term of 5,8,11,14,… .

STEP-BY-STEP SOLUTION

Difference is 3. Multiples of 3 are 3,6,9,12,…; add 2. Rule: 3n + 2.

Worked example 2

Find the nth term of 12,7,2,−3,… .

STEP-BY-STEP SOLUTION

Difference is −5. Using −5n + c, the first term gives −5 + c = 12, so c = 17. Rule: 17 − 5n.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

The common difference is not normally the whole nth-term formula. For 5,8,11,..., writing 3n gives 3,6,9,... . A constant shift is also needed.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Find the nth term of 4,9,14,19,... .

2.2 Find the nth term of 10,8,6,4,... .

2.3 Find the 30th term of 4n + 7.

2.4 Find the nth term of 1/2,1,3/2,2,... .

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Membership, missing terms and number patterns

To test whether a number belongs to a sequence, set the nth-term expression equal to it and solve for n. Membership requires a permitted position, usually a positive integer. A fractional or non-positive solution does not identify a term in a sequence beginning at n = 1.

Consecutive triangular numbers increase by 2,3,4,... after the first term, while square numbers increase by successive odd numbers. These patterns are not linear because their first differences change. Distinguishing the pattern matters more than forcing every sequence into dn + c.

A reliable method

1. Set the rule equal to the proposed term.

2. Solve for n and check its domain.

3. Inspect differences before deciding a sequence is linear.

Worked example 1

Is 83 a term of 5n + 3?

STEP-BY-STEP SOLUTION

5n + 3 = 83 gives n = 16, a positive integer. Yes, it is the 16th term.

Worked example 2

Is 42 a term of 4n + 1?

STEP-BY-STEP SOLUTION

4n + 1 = 42 gives n = 41/4 = 10.25. This is not an integer position, so no.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A formula can produce values at n = 0 or n = 1/2 algebraically, but those inputs do not belong to a sequence indexed by positive integers. Domain is part of the rule.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Is 61 a term of 6n + 1?

3.2 Is 50 a term of 3n + 2?

3.3 Find the missing term in a linear sequence 8, __, 20, 26.

3.4 Continue the triangular numbers 1,3,6,10,15 by two terms.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Write the next three terms after 4,9,14,19 using the same linear rule. [1 mark]

R2. Find the first three terms of $7n - 3$. [2 marks]

R3. Find the nth term of 6,10,14,18,… . [2 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Find the nth term of 15,12,9,6,... . [2 marks]

R5. Is 100 a term of 7n + 2? Justify. [3 marks]

R6. A row pattern has 5 tiles in row 1 and 3 extra tiles in each next row. Find row 20. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Generating terms and recognising patterns

1.1 Start at −2 and add 5. Write five terms.

−2, 3, 8, 13, 18.

1.2 Find the first four terms of 3n − 1.

2, 5, 8, 11.

1.3 A Fibonacci-type sequence starts 2,5. Write the next four terms.

7, 12, 19, 31.

1.4 Write the first five positive square numbers.

1, 4, 9, 16, 25.

Lesson 2 • Finding a linear nth term

2.1 Find the nth term of 4,9,14,19,… .

5n − 1.

2.2 Find the nth term of 10,8,6,4,… .

12 − 2n.

2.3 Find the 30th term of 4n + 7.

4 × 30 + 7 = 127.

2.4 Find the nth term of 1/2,1,3/2,2,… .

n/2.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Membership, missing terms and number patterns

3.1 Is 61 a term of 6n + 1?

6n = 60, so n = 10: yes.

3.2 Is 50 a term of 3n + 2?

3n = 48, so n = 16: yes.

3.3 Find the missing term in a linear sequence 8, __, 20, 26.

The difference is 6, so the missing term is 14.

3.4 Continue the triangular numbers 1,3,6,10,15 by two terms.

Add 6 then 7: 21, 28.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Write the next three terms after 4,9,14,19 using the same linear rule. [1 mark]

24, 29, 34. [1 for all three.]

R2. Find the first three terms of $7n - 3$. [2 marks]

4, 11, 18. [1 for substitution method; 1 for all terms.]

R3. Find the nth term of 6,10,14,18,… . [2 marks]

Difference 4, offset 2: $4n + 2$. [1 for 4n; 1 for constant.]

R4. Find the nth term of 15,12,9,6,… . [2 marks]

$18 - 3n$. [1 for difference; 1 for offset.]

R5. Is 100 a term of $7n + 2$? Justify. [3 marks]

$7n + 2 = 100$ gives $n = 14$, so yes, the 14th term. [1 for equation; 1 for n; 1 for conclusion.]

R6. A row pattern has 5 tiles in row 1 and 3 extra tiles in each next row. Find row 20. [3 marks]

Rule $3n + 2$; at $n = 20$ it gives 62 tiles. [1 for difference; 1 for rule; 1 for output.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 20

20

Quadratic and geometric sequences

Higher-level GCSE material

Changing differences and repeated multiplication produce patterns beyond linear sequences. Develop quadratic nth terms systematically and use geometric rules with powers, including fractional ratios.

$an^2 + bn + c$ • second difference $2a$

Quadratic differences and geometric multipliers.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Find linear nth terms, use powers and substitute into expressions.

Your route through this notebook

1. Recognising quadratic sequences
2. Finding a quadratic nth term
3. Geometric sequences and exponential rules

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Recognising quadratic sequences

A quadratic sequence has nth term $an^2 + bn + c$ with $a \neq 0$. Its first differences change, but its second differences are constant. For consecutive integer positions, the constant second difference is $2a$. This gives a direct connection between a number pattern and the coefficient of n^2.

A constant second difference is evidence for the intended quadratic pattern, not proof that every possible continuation of a finite list must be quadratic. Check the supplied terms and context. Negative second differences produce negative leading coefficients, so the sequence can eventually turn downwards.

A reliable method

1. Find first differences, then differences of those differences.

2. Use a = half the constant second difference.

3. Compare with an^2 to find the remaining linear part.

Worked example 1

Find the second difference of 3,8,15,24,35.

STEP-BY-STEP SOLUTION

First differences: 5,7,9,11. Second differences: 2,2,2. Therefore the intended quadratic has $a = 1$.

Worked example 2

A quadratic sequence has second difference 6. What is its n^2 coefficient?

STEP-BY-STEP SOLUTION

Since $2a = 6$, $a = 3$. The remaining coefficients still require term information.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A constant first difference signals a linear sequence; a non-zero constant second difference signals a quadratic pattern. Do not use a linear formula merely because the first few terms increase.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Find the first and second differences of 2,7,14,23,34.

1.2 Find a when the second difference is 8.

1.3 Find a when the second difference is −4.

1.4 Continue 1,4,9,16,25 using the square-number rule.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Finding a quadratic nth term

After finding a, subtract the sequence an^2 term by term. The remainders form a linear sequence $bn + c$. Find its nth term using the constant difference method, then add an^2 back. This separates the changing part of the differences from the remaining constant shift and linear growth.

Alternatively, substitute $n = 1,2,3$ into $an^2 + bn + c$ and solve the resulting equations. The subtraction method is often clearer for short GCSE sequences. Check the final formula against at least three supplied terms.

A reliable method

1. Find a from second differences.

2. Subtract an^2 at matching positions.

3. Find the linear remainder and combine the expressions.

Worked example 1

Find the nth term of 4,9,16,25,36.

STEP-BY-STEP SOLUTION

Second difference 2 gives $a = 1$. Subtract 1,4,9,16,25 to leave 3,5,7,9,11, whose rule is $2n + 1$. Result: $n^2 + 2n + 1$.

Worked example 2

Find the nth term of 5,14,29,50.

STEP-BY-STEP SOLUTION

Second differences are 6, so $a = 3$. Subtract 3,12,27,48 to leave 2,2,2,2. Result: $3n^2 + 2$.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Subtract corresponding positions. The first comparison uses a $\times 1^2$, not a $\times 0^2$, when the sequence begins at $n = 1$.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Find the nth term of 2,5,10,17,26.

2.2 Find the nth term of 3,10,21,36.

2.3 Find the 12th term of $n^2 + 3n - 2$.

2.4 Is 50 in the sequence $n^2 + 1$, $n \geq 1$?

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Geometric sequences and exponential rules

A geometric sequence multiplies by a fixed ratio r to get the next term. If its first term is a, the nth term is ar^{n-1}. The exponent is n − 1 because reaching the first term requires no multiplications by r. With a positive ratio below 1, positive terms decrease; with ratio greater than 1, they grow.

Square-root ratios work exactly like other ratios. Some sequences use negative ratios and alternate signs; recognise them even when the focus is positive-ratio GCSE patterns. A recursive rule and a direct nth-term rule are different descriptions of the same sequence.

A reliable method

1. Divide consecutive non-zero terms to find the ratio.

2. Use first term × ratio^{n-1}.

3. Check n = 1 and n = 2 before using a distant position.

Worked example 1

Find the nth term of 3,6,12,24,... .

STEP-BY-STEP SOLUTION

First term 3, ratio 2. Rule: $3 \times 2^{n-1}$. The 5th term is 3 × 16 = 48.

Worked example 2

Find the 6th term of 80,40,20,10,... .

STEP-BY-STEP SOLUTION

The ratio is 1/2. Term 6 = $80 \times (1/2)^5 = 2.5$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A geometric sequence is not recognised by equal differences. For 2,6,18,54 the ratios are all 3 while the differences are 4,12,36. Compare the correct relationship.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Find the nth term of 5,15,45,135,... .

3.2 Find the 5th term of 48,24,12,... .

3.3 Find the ratio of $2,2\sqrt{3},6,6\sqrt{3},\ldots$.

3.4 Find the first four terms of $4 \times 3^{n-1}$.

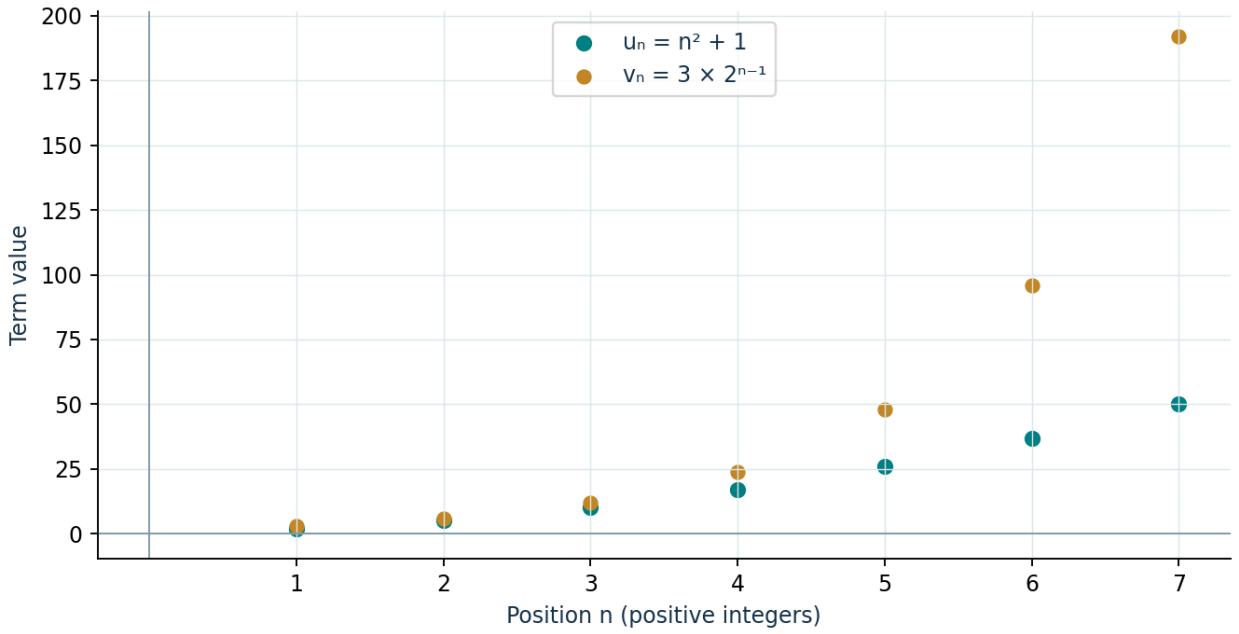
Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

Position is discrete: plot terms as points



WHAT THE FIGURE SHOWS

These are sequences indexed by positive integers, so the plotted terms are separate points. The quadratic rule adds changing differences; the geometric rule doubles each term. The geometric values eventually overtake the quadratic values.

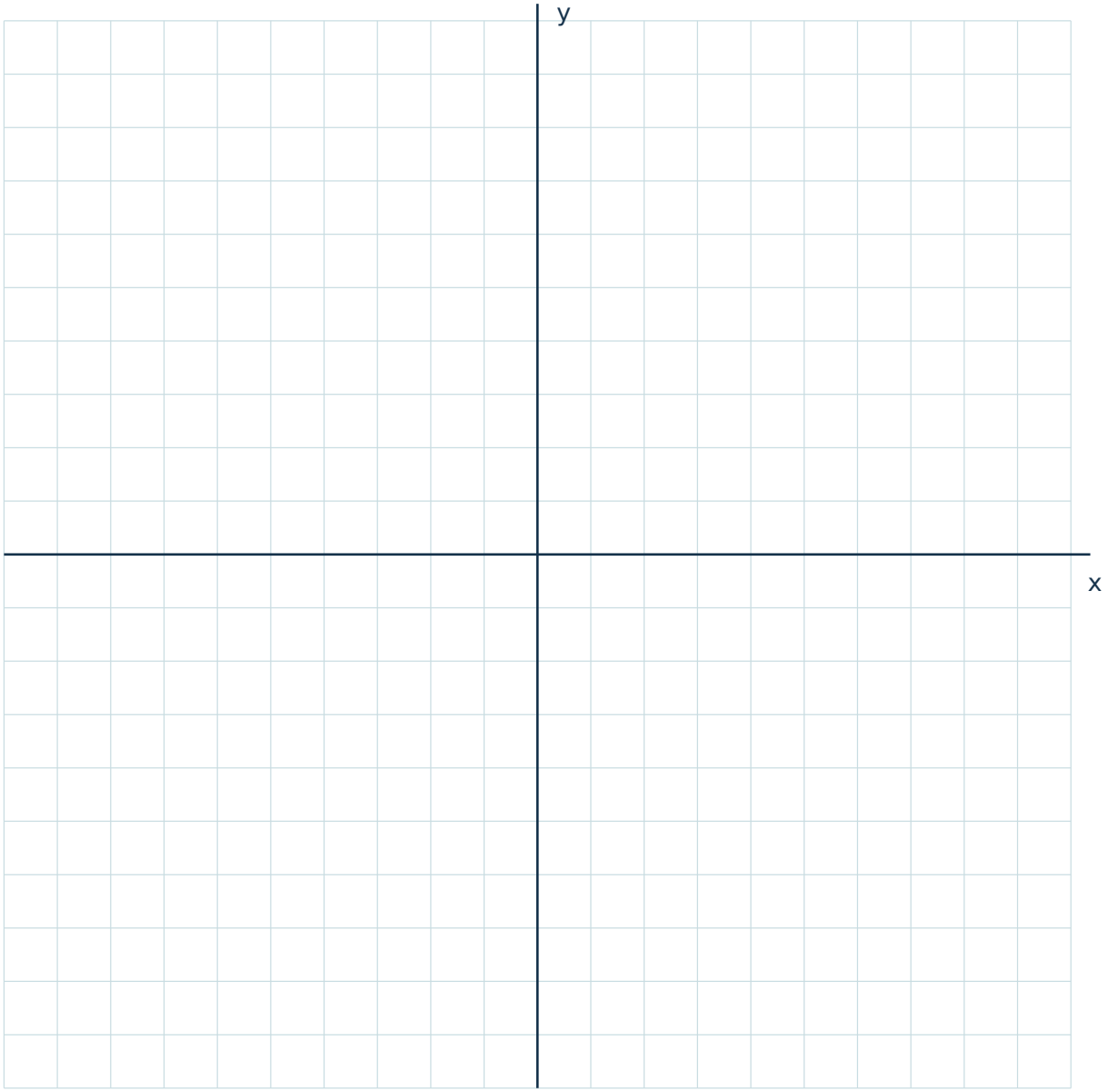
Explain how the labelled features follow from the equations. Check a point by substitution or recalculate a gradient or area before relying on the picture.

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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

Choose one graph from the preceding visual explainer and reconstruct it on this grid. Choose and label a suitable scale on each axis; one square need not represent one unit. Include the displayed domain, key points and units. For a real-life graph, relabel x and y with the measured quantities.



SELF-CHECK

Compare with the preceding plotted figure and its explanation. Check at least three points, any intercepts and the direction or shape of the graph. Explain how the equation or context produces those features.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Find the second difference of 1,6,15,28,45. [2 marks]

R2. Find the nth term of 2,8,18,32,50. [3 marks]

R3. Find the nth term of 6,11,18,27,38. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Find term 15 of $2n^2 - n + 4$. [2 marks]

R5. Find the nth term of 7,21,63,... . [2 marks]

R6. Find the 8th term of 128,64,32,... . [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Recognising quadratic sequences

1.1 Find the first and second differences of 2,7,14,23,34.

First: 5,7,9,11. Second: 2,2,2.

1.2 Find a when the second difference is 8.

$a = 4$.

1.3 Find a when the second difference is -4.

$a = -2$.

1.4 Continue 1,4,9,16,25 using the square-number rule.

36, 49, corresponding to 6^2 and 7^2.

Lesson 2 • Finding a quadratic nth term

2.1 Find the nth term of 2,5,10,17,26.

Subtract n^2: 1,1,1,1,1. Rule: $n^2 + 1$.

2.2 Find the nth term of 3,10,21,36.

Second difference 4 gives $2n^2$. Remainder 1,2,3,4 gives n. Rule: $2n^2 + n$.

2.3 Find the 12th term of $n^2 + 3n - 2$.

$144 + 36 - 2 = 178$.

2.4 Is 50 in the sequence $n^2 + 1$, $n \geq 1$?

$n^2 + 1 = 50$ gives $n^2 = 49$ and positive position $n = 7$. Yes.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Geometric sequences and exponential rules

3.1 Find the nth term of 5,15,45,135,... .

$5 \times 3^{n-1}$.

3.2 Find the 5th term of 48,24,12,... .

$48 \times (1/2)^4 = 3$.

3.3 Find the ratio of $2,2\sqrt{3},6,6\sqrt{3},\ldots$.

The common ratio is $\sqrt{3}$.

3.4 Find the first four terms of $4 \times 3^{n-1}$.

4,12,36,108.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Find the second difference of 1,6,15,28,45. [2 marks]

First differences 5,9,13,17; second difference 4. [1 for each stage.]

R2. Find the nth term of 2,8,18,32,50. [3 marks]

Second difference 4 gives $2n^2$; there is no remainder. [1 for differences; 1 for coefficient; 1 for rule.]

R3. Find the nth term of 6,11,18,27,38. [3 marks]

Second difference 2; subtract n^2 to leave 5,7,9,11,13 = 2n + 3. Rule $n^2 + 2n + 3$. [1 for coefficient; 1 for remainder; 1 for rule.]

R4. Find term 15 of $2n^2 - n + 4$. [2 marks]

$2 \times 225 - 15 + 4 = 439$. [1 for substitution; 1 for result.]

R5. Find the nth term of 7,21,63,… . [2 marks]

$7 \times 3^{n-1}$. [1 for ratio; 1 for correct exponent and first term.]

R6. Find the 8th term of 128,64,32,… . [3 marks]

$128 \times (1/2)^7 = 1$. [1 for ratio; 1 for exponent; 1 for result.]

Reflect, repair and revisit

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 21

21

Coordinates and straight-line graphs

Shared core with Higher extensions

Coordinates connect algebra to position. Learn to plot lines, interpret gradient and intercept, and find equations from points. Higher extensions include perpendicular lines. Use equal coordinate units and read every scale carefully.

$$y = mx + c$$

Gradient m measures change; c is the intercept.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

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Before you begin

Substitute values, solve linear equations and calculate with negative numbers.

Your route through this notebook

1. Coordinates and tables of values
2. Gradient, intercept and line equations
3. Parallel and perpendicular lines — extension

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Coordinates and tables of values

An ordered pair (x,y) gives horizontal position first and vertical position second. The origin is (0,0). Negative x is left of the vertical axis and negative y is below the horizontal axis. The order matters: (2,5) and (5,2) usually name different points.

For a graph y = f(x), choose x values and calculate the corresponding y values. Plot the pairs accurately. A straight-line rule gives collinear points, so use a ruler and extend the line as appropriate. At least a third point helps check an error when two points alone would always define some line.

A reliable method

1. Read x horizontally, then y vertically.

2. Prepare a table with substitutions shown where signs are difficult.

3. Plot accurately and check the shape against the rule.

Worked example 1

Find points on $y = 2x - 1$ for $x = -1,0,2$.

STEP-BY-STEP SOLUTION

Substitution gives $y = -3,-1,3$. Plot $(-1,-3)$, $(0,-1)$, $(2,3)$.

Worked example 2

Find the midpoint of $(-2,4)$ and $(6,10)$.

STEP-BY-STEP SOLUTION

Average coordinates: $((-2 + 6)/2, (4 + 10)/2) = (2,7)$.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

On a graph, one small square need not mean one unit. Read labelled ticks on each axis separately; horizontal and vertical scales may differ.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Give the coordinates of a point 3 left and 5 up from the origin.

1.2 For $y = 3x + 2$, find y at $x = -2,0,2$.

1.3 Does (4,9) lie on $y = 2x + 1$?

1.4 Find the midpoint of $(1,-3)$ and (7,5).

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Gradient, intercept and line equations

In $y = mx + c$, m is the gradient and c is the y-intercept. Gradient is vertical change divided by horizontal change, using the same order of points in both differences. A positive gradient rises left to right; a negative gradient falls. A horizontal line has gradient zero. A vertical line has equation $x = k$ and no finite gradient.

To find a line from two points, calculate m, substitute one point into $y = mx + c$ and solve for c. The y-intercept is the y value at $x = 0$, not an arbitrary point where the line crosses another graph.

A reliable method

1. Calculate $m = (y_2 - y_1)/(x_2 - x_1)$.

2. Substitute a known point to find c.

3. Check the second point in the completed equation.

Worked example 1

Find the line through (1,5) and (4,11).

STEP-BY-STEP SOLUTION

$m = (11 - 5)/(4 - 1) = 2$. Use $5 = 2 \times 1 + c$, so $c = 3$. Equation: $y = 2x + 3$.

Worked example 2

Find the x-intercept of $y = -3x + 12$.

STEP-BY-STEP SOLUTION

At the x-axis $y = 0$. Solve $0 = -3x + 12$ to get $x = 4$. Intercept (4,0).

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Gradient is a ratio of coordinate changes, not the visual angle on a distorted display. A rise of 4 for a run of 2 gives gradient 2, whatever the printed scale.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 State gradient and y-intercept of $y = -4x + 7$.

2.2 Find the gradient through $(-1,2)$ and $(3,10)$.

2.3 Find the line of gradient 3 through $(2,8)$.

2.4 Write the horizontal line through $(5,-2)$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Parallel and perpendicular lines — extension

Distinct non-vertical parallel lines have equal gradients and different intercepts. A line through a given point parallel to another line uses the same gradient; substitute the point to find its intercept. Coincident lines have both the same gradient and the same intercept.

For non-zero finite gradients, perpendicular lines have gradients whose product is −1. Therefore a perpendicular gradient is the negative reciprocal. A line of gradient 2 is perpendicular to one of gradient −1/2. Horizontal and vertical lines form a separate perpendicular pair. These statements assume the usual equal-unit Cartesian geometry.

A reliable method

1. Put a line into y = mx + c before reading its gradient.

2. For parallel lines keep m; for perpendicular lines use −1/m.

3. Use the supplied point to determine the new intercept.

Worked example 1

Find a line parallel to y = 4x − 1 through (2,11).

STEP-BY-STEP SOLUTION

Use y = 4x + c. Then 11 = 8 + c gives c = 3. Result: y = 4x + 3.

Worked example 2

Higher: find a line perpendicular to y = 2x + 5 through (4,1).

STEP-BY-STEP SOLUTION

New gradient −1/2. In 1 = (−1/2)4 + c, c = 3. Result: y = −x/2 + 3.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

The perpendicular gradient is not merely the negative gradient. For m = 3 it is −1/3, not −3. Multiply the two gradients to check for −1.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Find a line parallel to y = −2x + 6 through (1,3).

3.2 Higher: find the perpendicular gradient to 5.

3.3 Higher: find the perpendicular gradient to −2/3.

3.4 Higher: find the line perpendicular to y = −x/3 + 4 through (2,1).

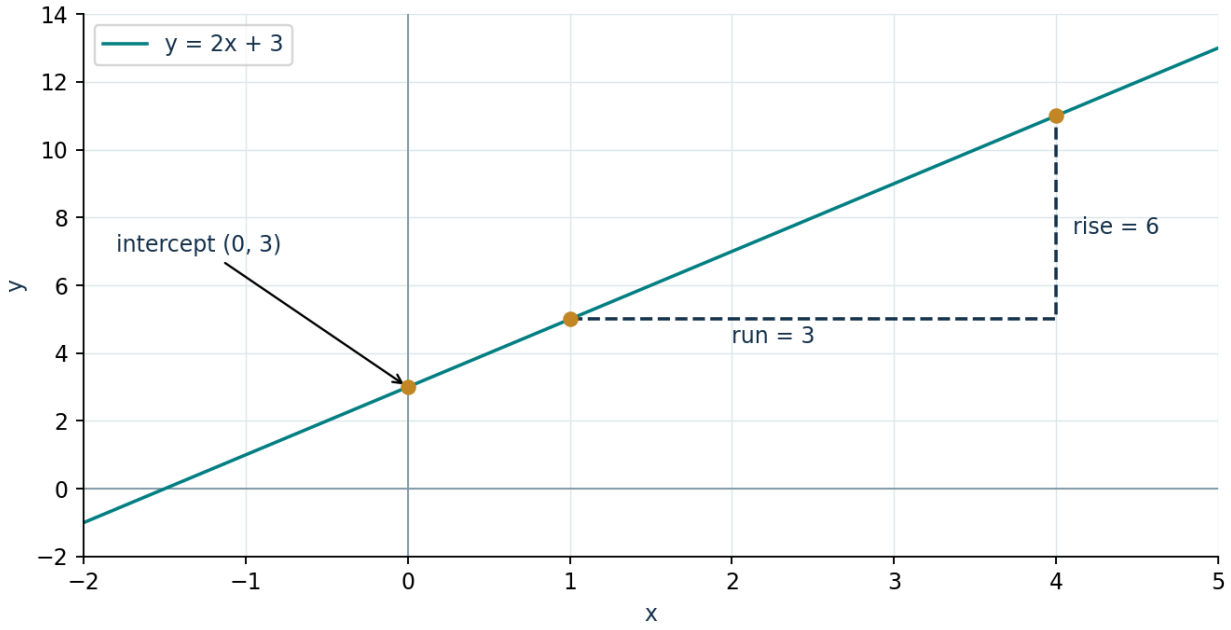
Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

Gradient and intercept tell the line’s story



WHAT THE FIGURE SHOWS

The line passes through (1,5) and (4,11). Its gradient is rise/run = 6/3 = 2, and its y-intercept is (0,3). The dashed triangle measures coordinate changes; it is not a second graph.

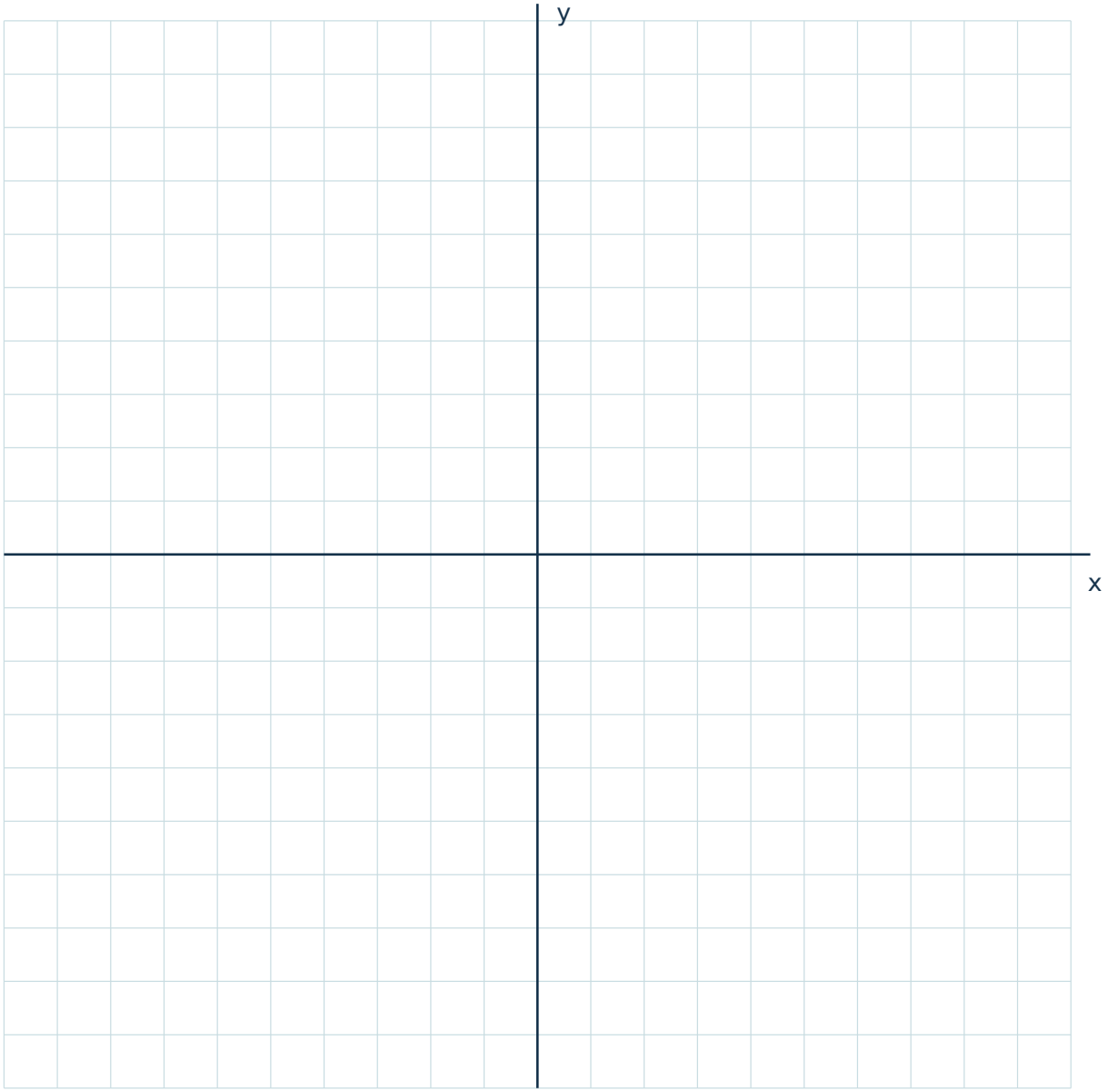
Explain how the labelled features follow from the equations. Check a point by substitution or recalculate a gradient or area before relying on the picture.

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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

Choose one graph from the preceding visual explainer and reconstruct it on this grid. Choose and label a suitable scale on each axis; one square need not represent one unit. Include the displayed domain, key points and units. For a real-life graph, relabel x and y with the measured quantities.



SELF-CHECK

Compare with the preceding plotted figure and its explanation. Check at least three points, any intercepts and the direction or shape of the graph. Explain how the equation or context produces those features.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Find y on $y = 4x - 3$ at $x = -2$. [2 marks]

R2. Find the gradient through (2,1) and (6,9). [2 marks]

R3. Find the equation of the line through (0,−2) and (3,7). [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Find the x-intercept of $y = 5x - 15$. [2 marks]

R5. Find a parallel line to $y = 2x + 7$ through (3,4). [3 marks]

R6. Higher: find the perpendicular line to $y = 4x - 1$ through (4,2). [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Coordinates and tables of values

1.1 Give the coordinates of a point 3 left and 5 up from the origin.

$(-3,5)$.

1.2 For $y = 3x + 2$, find y at $x = -2,0,2$.

$-4,2,8$, giving $(-2,-4)$, $(0,2)$, $(2,8)$.

1.3 Does $(4,9)$ lie on $y = 2x + 1$?

Yes: $2 \times 4 + 1 = 9$.

1.4 Find the midpoint of $(1,-3)$ and $(7,5)$.

$(4,1)$.

Lesson 2 • Gradient, intercept and line equations

2.1 State gradient and y-intercept of $y = -4x + 7$.

Gradient -4; y-intercept $(0,7)$.

2.2 Find the gradient through $(-1,2)$ and $(3,10)$.

$(10 - 2)/(3 - (-1)) = 8/4 = 2$.

2.3 Find the line of gradient 3 through $(2,8)$.

$8 = 3 \times 2 + c$ gives $c = 2$. Thus $y = 3x + 2$.

2.4 Write the horizontal line through $(5,-2)$.

$y = -2$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Parallel and perpendicular lines — extension

3.1 Find a line parallel to y = −2x + 6 through (1,3).

y = −2x + c; 3 = −2 + c, so y = −2x + 5.

3.2 Higher: find the perpendicular gradient to 5.

−1/5.

3.3 Higher: find the perpendicular gradient to −2/3.

3/2.

3.4 Higher: find the line perpendicular to y = −x/3 + 4 through (2,1).

New gradient 3. Then 1 = 6 + c, so y = 3x − 5.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Find y on $y = 4x - 3$ at $x = -2$. [2 marks]

$y = -8 - 3 = -11$. [1 for substitution; 1 for result.]

R2. Find the gradient through (2,1) and (6,9). [2 marks]

$(9 - 1)/(6 - 2) = 2$. [1 for differences; 1 for quotient.]

R3. Find the equation of the line through (0,−2) and (3,7). [3 marks]

Gradient $9/3 = 3$ and intercept -2, so $y = 3x - 2$. [1 for gradient; 1 for intercept; 1 for equation.]

R4. Find the x-intercept of $y = 5x - 15$. [2 marks]

Set $y = 0$: $x = 3$, giving (3,0). [1 for setting zero; 1 for point.]

R5. Find a parallel line to $y = 2x + 7$ through (3,4). [3 marks]

$4 = 2 \times 3 + c$ gives $c = -2$, so $y = 2x - 2$. [1 for gradient; 1 for c; 1 for equation.]

R6. Higher: find the perpendicular line to $y = 4x - 1$ through (4,2). [3 marks]

Gradient $-1/4$; $2 = -1 + c$ gives $c = 3$. Thus $y = -x/4 + 3$. [1 for gradient; 1 for c; 1 for equation.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 22

22

Quadratic graphs

Shared core with Higher extensions

A quadratic graph is a parabola. Connect tables and sketches with roots, intercepts and symmetry, then use completed-square form to identify turning points. Actual plotted examples accompany the algebra.

$$y = (x - 3)^2 - 4$$

Roots, intercepts and the turning point fit together.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

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Before you begin

Substitute signed numbers, expand brackets and factorise simple quadratics.

Your route through this notebook

1. Tables, plotting and parabolic shape
2. Roots, intercepts and graphical solutions
3. Turning points and completed-square form — Higher

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Tables, plotting and parabolic shape

A quadratic has form $y = ax^2 + bx + c$ with $a \neq 0$. Its graph is a smooth parabola, opening upwards if $a > 0$ and downwards if $a < 0$. Use brackets when squaring negative x values. Plot enough points on both sides of the turning point to show the shape.

A curve should pass smoothly through accurate points rather than zigzag between them. Equal x values either side of the symmetry line produce equal y values. A restricted x interval shows only part of the graph, so do not infer the full range from a short table alone.

A reliable method

1. Choose x values covering both sides of the vertex.

2. Calculate each y using the original rule.

3. Draw a smooth curve and check symmetry.

Worked example 1

Make a table for $y = x^2 - 4$ at $x = -2,-1,0,1,2$.

STEP-BY-STEP SOLUTION

The y values are $0,-3,-4,-3,0$. The minimum is $(0,-4)$ and the graph is symmetric about $x = 0$.

Worked example 2

Describe $y = -2x^2 + 3$.

STEP-BY-STEP SOLUTION

The negative leading coefficient makes it open downwards. It crosses the y-axis at $(0,3)$, which is also its maximum point.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

For $x = -2$, x^2 is 4, not -4. A missing pair of substitution brackets can turn an upward parabola into an incorrect table.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Find $y = x^2 + 1$ at $x = -2, -1, 0, 1, 2$.

1.2 State the opening direction of $y = 3x^2 - 2$.

1.3 State the y-intercept of $y = x^2 - 5x + 6$.

1.4 Find the symmetry line of $y = (x - 2)^2 + 1$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Roots, intercepts and graphical solutions

Roots are x values where y = 0, so they are the x-coordinates of x-axis intersections. Factorisation can identify exact roots; a graph can give approximate readings. The y-intercept is found by substituting x = 0. These are different kinds of intercept.

To solve $x^2 - 4 = 5$ using the graph $y = x^2 - 4$, draw the horizontal line y = 5 and read the intersection x values. To solve two graph rules simultaneously, read their intersections. A tangent touching the axis gives a repeated root; a graph missing the axis has no real roots.

A reliable method

1. Set y = 0 for roots, and x = 0 for the y-intercept.
2. For y = k, use the horizontal line at height k.
3. State graph readings to an accuracy justified by the scale.

Worked example 1

Find roots and y-intercept of $y = x^2 - 5x + 6$.

STEP-BY-STEP SOLUTION

Factorise $(x - 2)(x - 3)$, giving roots 2 and 3. The y-intercept is (0,6).

Worked example 2

Use $y = x^2 - 4$ to solve $x^2 - 4 = 5$ exactly.

STEP-BY-STEP SOLUTION

Intersections with y = 5 satisfy $x^2 = 9$, so $x = -3$ or 3. A plotted graph should show these two intersections.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A root is usually reported as an x value; its coordinate point is (x,0). Do not describe the y-intercept (0,c) as a root unless c is zero.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Find roots of $y = x^2 - 9$.

2.2 Find roots of $y = (x + 2)(x - 5)$.

2.3 How many real roots has $y = x^2 + 4$?

2.4 Solve $x^2 - 1 = 8$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Turning points and completed-square form — Higher

In $y = a(x - h)^2 + k$, the squared expression is zero at $x = h$. The turning point is therefore (h,k), with symmetry line $x = h$. For $a > 0$ it is a minimum; for $a < 0$ it is a maximum. The sign inside the bracket is opposite the coordinate h.

Completing the square rewrites an expanded quadratic into this revealing form. For $x^2 + bx$, take half of b and square it: $x^2 + bx = (x + b/2)^2 - (b/2)^2$. Add the original constant afterwards. This also helps find roots and the range of y values.

A reliable method

1. Read or create $a(x - h)^2 + k$.

2. State the turning point and symmetry line.

3. Use the sign of a to identify a minimum or maximum.

Worked example 1

Find the turning point of $y = x^2 - 6x + 5$.

STEP-BY-STEP SOLUTION

Complete the square: $(x - 3)^2 - 9 + 5 = (x - 3)^2 - 4$. Minimum point $(3,-4)$; symmetry $x = 3$.

Worked example 2

Describe the turning point and range of $y = -2(x + 1)^2 + 7$.

STEP-BY-STEP SOLUTION

Turning point $(-1,7)$, a maximum. Since the squared term is non-negative, $y \leq 7$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

For $y = (x + 4)^2 - 2$, the turning point is $(-4,-2)$, not $(4,-2)$. Set the bracket equal to zero to decide its horizontal position.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Find the turning point of $y = (x - 5)^2 + 3$.

3.2 Complete the square for $x^2 + 4x + 1$.

3.3 Find the minimum of $y = x^2 + 8x + 10$.

3.4 Find the roots of $y = (x - 2)^2 - 9$.

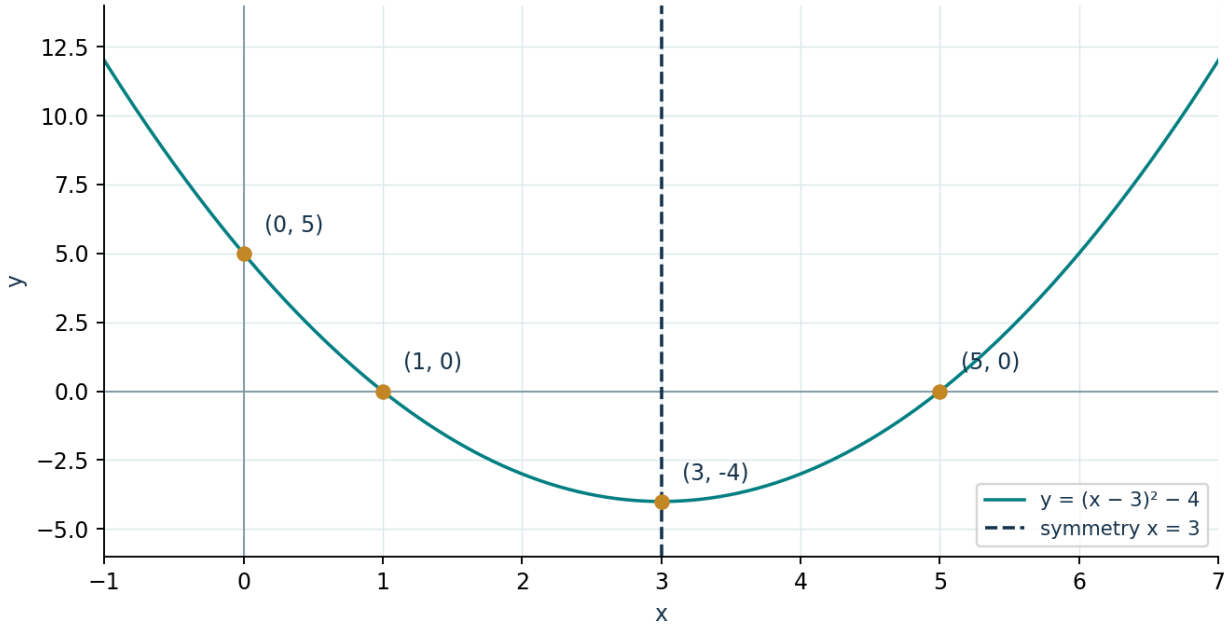
Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

Roots, turning point and symmetry



WHAT THE FIGURE SHOWS

Expanding gives $y = x^2 - 6x + 5$. Roots are 1 and 5; the minimum is $(3,-4)$; the y-intercept is $(0,5)$. Equal horizontal steps either side of $x = 3$ give equal heights. These features should agree with any table of values.

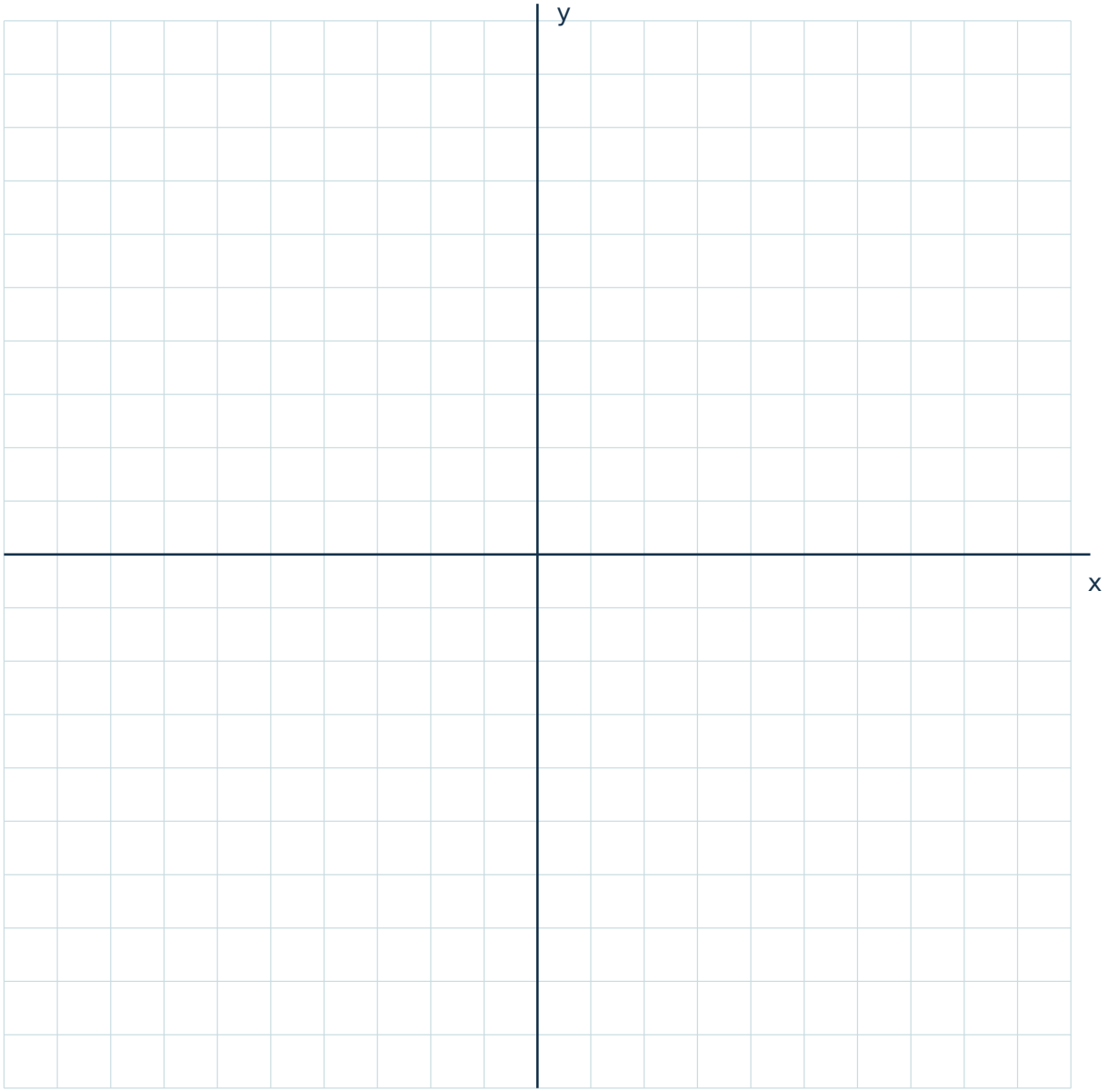
Explain how the labelled features follow from the equations. Check a point by substitution or recalculate a gradient or area before relying on the picture.

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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

Choose one graph from the preceding visual explainer and reconstruct it on this grid. Choose and label a suitable scale on each axis; one square need not represent one unit. Include the displayed domain, key points and units. For a real-life graph, relabel x and y with the measured quantities.



SELF-CHECK

Compare with the preceding plotted figure and its explanation. Check at least three points, any intercepts and the direction or shape of the graph. Explain how the equation or context produces those features.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Calculate $y = x^2 - 2x$ for $x = -2$. [2 marks]

R2. Find the y-intercept of $y = 2x^2 + 3x - 5$. [1 mark]

R3. Find roots of $y = x^2 - 7x + 10$. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Solve $x^2 - 4 = 12$. [2 marks]

R5. Higher: complete the square for $x^2 - 10x + 21$. [3 marks]

R6. Higher: state the turning point, symmetry line and range of $y = -(x - 3)^2 + 8$. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Tables, plotting and parabolic shape

1.1 Find $y = x^2 + 1$ at $x = -2,-1,0,1,2$.

5,2,1,2,5.

1.2 State the opening direction of $y = 3x^2 - 2$.

Upwards, because $3 > 0$.

1.3 State the y-intercept of $y = x^2 - 5x + 6$.

At $x = 0$, $y = 6$, so (0,6).

1.4 Find the symmetry line of $y = (x - 2)^2 + 1$.

$x = 2$.

Lesson 2 • Roots, intercepts and graphical solutions

2.1 Find roots of $y = x^2 - 9$.

$x = -3$ and 3.

2.2 Find roots of $y = (x + 2)(x - 5)$.

$x = -2$ and 5.

2.3 How many real roots has $y = x^2 + 4$?

None: its minimum y is 4, above the x-axis.

2.4 Solve $x^2 - 1 = 8$.

$x^2 = 9$, so $x = -3$ or 3.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Turning points and completed-square form — Higher

3.1 Find the turning point of $y = (x - 5)^2 + 3$.

(5,3), a minimum.

3.2 Complete the square for $x^2 + 4x + 1$.

$(x + 2)^2 - 3$.

3.3 Find the minimum of $y = x^2 + 8x + 10$.

$y = (x + 4)^2 - 6$, so the minimum value is -6 at $x = -4$.

3.4 Find the roots of $y = (x - 2)^2 - 9$.

$(x - 2)^2 = 9$, so $x - 2 = \pm 3$. Roots -1 and 5.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Calculate $y = x^2 - 2x$ for $x = -2$. [2 marks]

$4 + 4 = 8$. [1 for substitution/signs; 1 for result.]

R2. Find the y-intercept of $y = 2x^2 + 3x - 5$. [1 mark]

$(0,-5)$. [1 for point.]

R3. Find roots of $y = x^2 - 7x + 10$. [3 marks]

$(x - 2)(x - 5) = 0$, so roots 2 and 5. [1 for factors; 1 for each root.]

R4. Solve $x^2 - 4 = 12$. [2 marks]

$x^2 = 16$, so $x = \pm 4$. [1 for isolation; 1 for both values.]

R5. Higher: complete the square for $x^2 - 10x + 21$. [3 marks]

$(x - 5)^2 - 4$. [1 for half coefficient; 1 for constant adjustment; 1 for final form.]

R6. Higher: state the turning point, symmetry line and range of $y = -(x - 3)^2 + 8$. [3 marks]

Maximum point (3,8), symmetry $x = 3$, range $y \leq 8$. [1 each.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 23

23

Cubic, reciprocal and exponential graphs

Shared recognition with Higher extensions

Different rules produce recognisable families of curves. Learn their key points, intercepts, symmetry and long-term behaviour. Pay attention to values that are excluded from a graph’s domain.

x^3 $\frac{1}{x}$ 2^x

Different rules produce different graph families.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Plot coordinates, calculate powers and substitute positive and negative values.

Your route through this notebook

1. Cubic graphs
2. Reciprocal curves and asymptotes
3. Exponential growth and decay — Higher extension

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Cubic graphs

The basic cubic $y = x^3$ passes through the origin and has rotational symmetry of order two about it. Negative inputs give negative outputs and positive inputs give positive outputs. Unlike $y = x^2$, its two ends point in opposite directions. The curve flattens at the origin but continues increasing through it.

Adding a constant translates the graph vertically. Multiplying by a negative coefficient reflects it in the x-axis and reverses its overall direction. More general cubic graphs can have turning points, so not every cubic looks exactly like x^3. For sketching, identify intercepts and useful points rather than reproducing a shape from memory without reference to the equation.

A reliable method

1. Calculate values on both sides of zero.
2. Identify intercepts and the sign of the leading term.
3. Draw a smooth curve with the correct end behaviour.

Worked example 1

Give points on $y = x^3$ for $x = -2,-1,0,1,2$.

STEP-BY-STEP SOLUTION

The points are $(-2,-8)$, $(-1,-1)$, $(0,0)$, $(1,1)$, $(2,8)$.

Worked example 2

Describe $y = -x^3 + 2$.

STEP-BY-STEP SOLUTION

Reflect $y = x^3$ in the x-axis, then translate up 2. Its y-intercept is $(0,2)$, and it decreases overall.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

An S-shaped cubic may have a stationary point without a maximum or minimum. At the origin, $y = x^3$ flattens but does not reverse direction.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Evaluate x^3 at $x = -3$.

1.2 Find the y-intercept of $y = x^3 - 5$.

1.3 Find the x-intercept of $y = x^3 - 8$.

1.4 Describe the symmetry of $y = x^3$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Reciprocal curves and asymptotes

For y = 1/x, x cannot be zero. Positive inputs give positive outputs and negative inputs give negative outputs, creating two separate branches. Near zero the magnitude of y grows very large; far from zero it approaches zero. The axes are asymptotes: lines approached by the curve but not reached in this basic example.

The graph y = k/x changes scale and, for negative k, changes the occupied quadrants. Do not join a line across x = 0. A plotted table with widely spaced values can hide the rapid change near the excluded input. Domain restrictions are part of the mathematics, not just a graphing inconvenience.

A reliable method

1. Exclude x = 0 and calculate points on both sides.

2. Sketch separate branches approaching the axes.

3. Use the sign of k to determine which quadrants contain the graph.

Worked example 1

Find points on y = 6/x for x = −3,−2,1,2,3.

STEP-BY-STEP SOLUTION

The points are (−3,−2), (−2,−3), (1,6), (2,3), (3,2).

Worked example 2

Describe the quadrants for y = −4/x.

STEP-BY-STEP SOLUTION

For positive x, y is negative; for negative x, y is positive. The branches are in quadrants IV and II.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

The fact that y approaches zero does not mean y = 0 is a point on y = 1/x. No finite non-zero x makes 1/x equal zero.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Find $y = 8/x$ at $x = -4$.

2.2 State the excluded input for $y = 3/x$.

2.3 State the asymptotes of $y = 5/x$.

2.4 Solve $6/x = 2$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Exponential growth and decay — Higher extension

In $y = a^x$, the input is in the exponent. For $a > 1$, values grow as x increases; for $0 < a < 1$ they decrease. Every positive-base graph passes through (0,1), because $a^0 = 1$. The outputs stay positive and the x-axis is an asymptote.

In $y = ka^x$, the initial value at $x = 0$ is k and a is a repeated multiplier for each unit increase in x. A 20% increase has multiplier 1.2; a 20% decrease has multiplier 0.8. This is multiplicative change, unlike the constant additive change of a straight line.

A reliable method

1. Identify the initial value and unit-step multiplier.
2. Calculate several inputs, including zero.
3. Distinguish exponential change from constant-rate linear change.

Worked example 1

Find $y = 2^x$ at $x = -2, -1, 0, 1, 2$.

STEP-BY-STEP SOLUTION

Values are 1/4, 1/2, 1, 2, 4. Negative exponents give reciprocals.

Worked example 2

A model is $P = 500(0.9)^t$. Interpret and evaluate at $t = 2$.

STEP-BY-STEP SOLUTION

Initial amount 500, with a 10% decrease per time unit. $P = 500 \times 0.81 = 405$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

$y = 2x$ and $y = 2^x$ are different rules. The first adds 2 to y for every unit of x; the second doubles y. They may share some points without being the same graph.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Find the y-intercept of $y = 3^x$.

3.2 Evaluate 4×2^x at $x = 3$.

3.3 Does $y = (1/2)^x$ grow or decay as x increases?

3.4 Write a model for an initial 200 growing by 5% each whole time step.

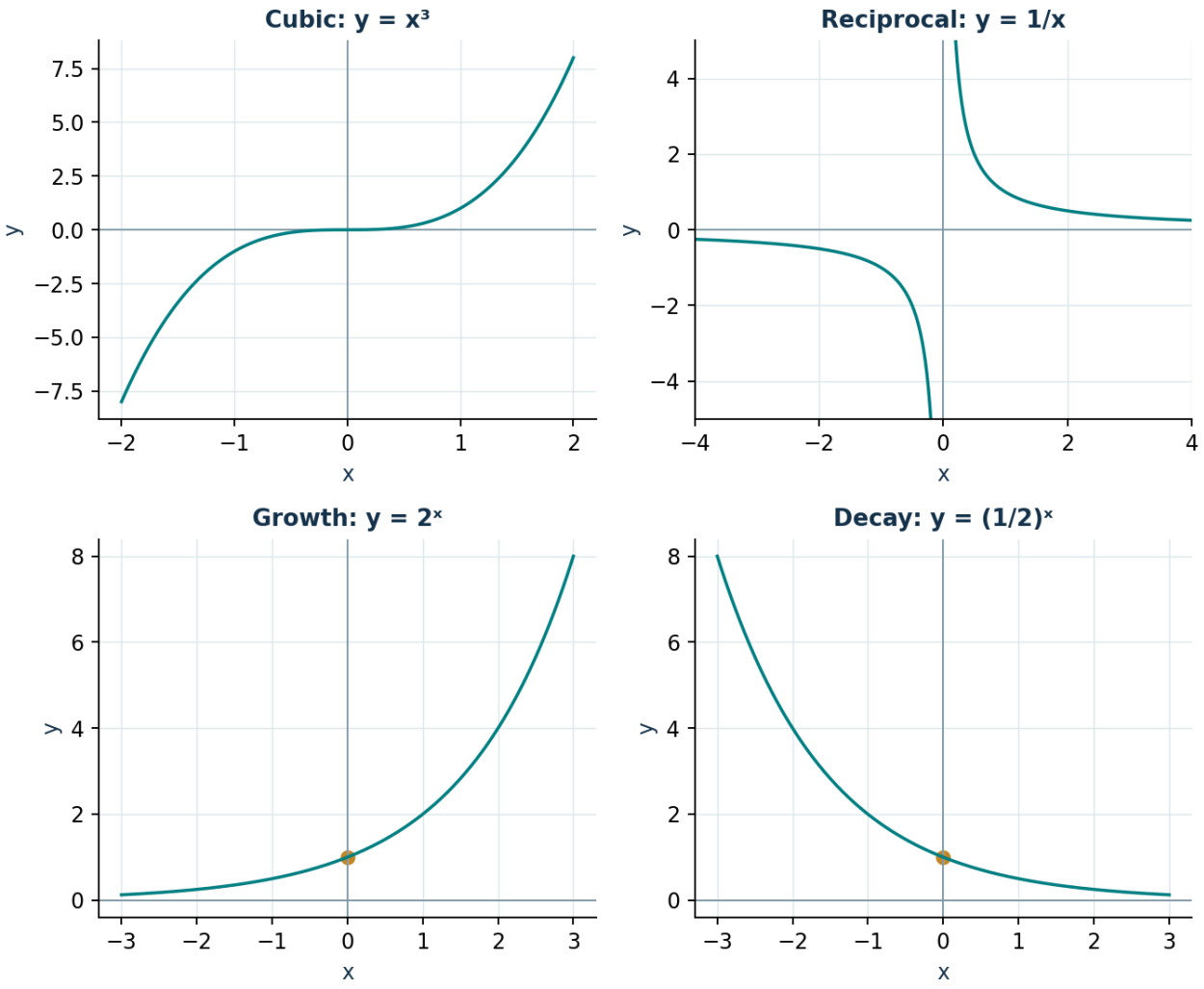
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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

Recognise families from equations and key features



WHAT THE FIGURE SHOWS

The cubic takes negative and positive values. The reciprocal has two branches and excludes x = 0. Both exponential curves pass through (0,1), stay positive and approach the x-axis. Growth and decay differ in whether the multiplier is above or below 1.

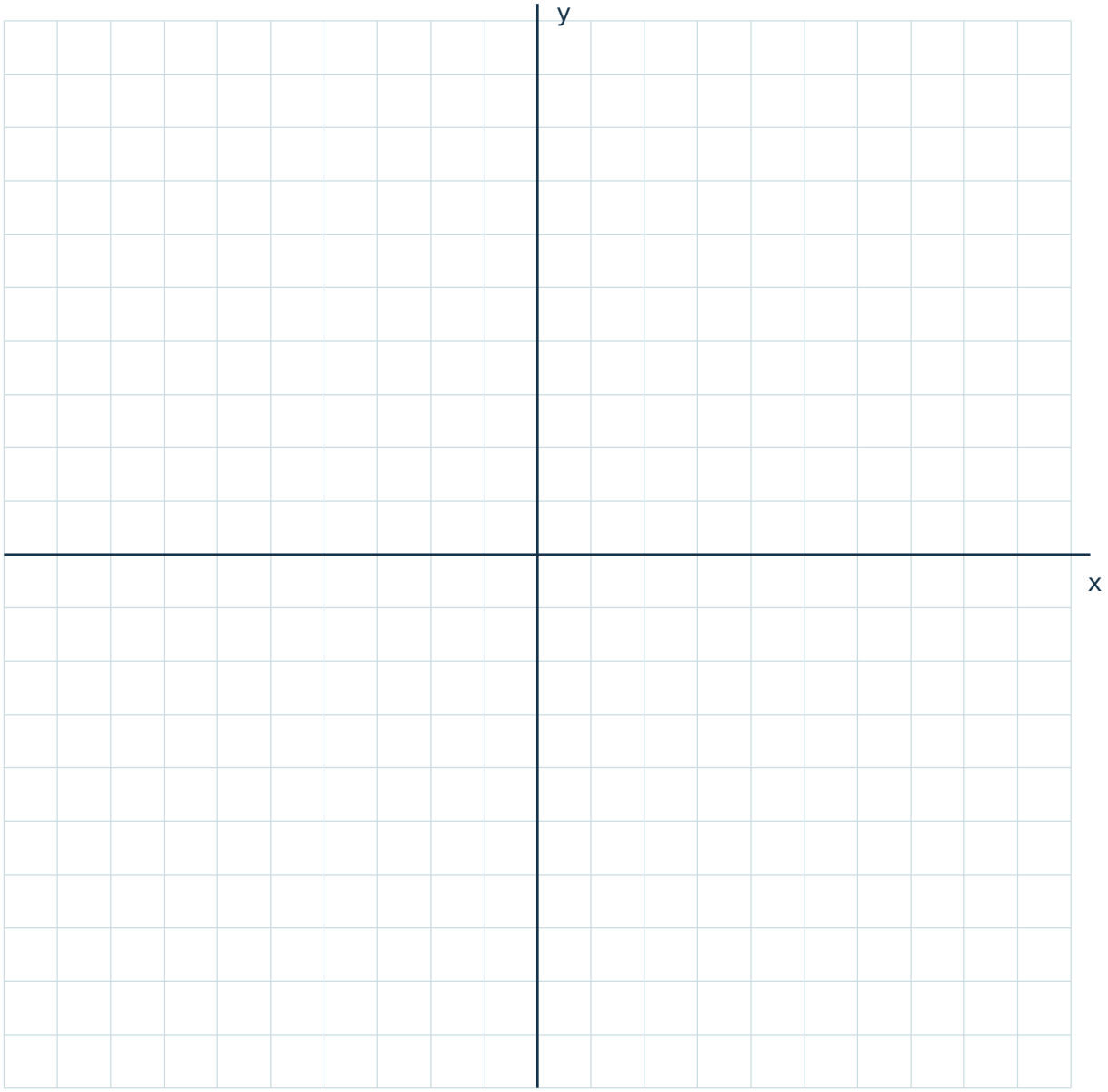
Explain how the labelled features follow from the equations. Check a point by substitution or recalculate a gradient or area before relying on the picture.

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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

Choose one graph from the preceding visual explainer and reconstruct it on this grid. Choose and label a suitable scale on each axis; one square need not represent one unit. Include the displayed domain, key points and units. For a real-life graph, relabel x and y with the measured quantities.



SELF-CHECK

Compare with the preceding plotted figure and its explanation. Check at least three points, any intercepts and the direction or shape of the graph. Explain how the equation or context produces those features.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Find $y = x^3 + 1$ at $x = -2$. [2 marks]

R2. Find the root of $y = x^3 + 27$. [2 marks]

R3. For $y = 12/x$, find y at $x = -6$ and state the excluded input. [2 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Explain why $y = 1/x$ has no x-intercept. [2 marks]

R5. Higher: evaluate $300(1.1)^2$. [2 marks]

R6. Higher: compare growth of $y = 3x$ and $y = 3^x$ at $x = 0,1,2$. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Cubic graphs

1.1 Evaluate x^3 at $x = -3$.

-27.

1.2 Find the y-intercept of $y = x^3 - 5$.

$(0,-5)$.

1.3 Find the x-intercept of $y = x^3 - 8$.

$x^3 = 8$ gives $x = 2$, so $(2,0)$.

1.4 Describe the symmetry of $y = x^3$.

Rotational symmetry of order two about the origin.

Lesson 2 • Reciprocal curves and asymptotes

2.1 Find $y = 8/x$ at $x = -4$.

-2.

2.2 State the excluded input for $y = 3/x$.

$x = 0$.

2.3 State the asymptotes of $y = 5/x$.

$x = 0$ and $y = 0$.

2.4 Solve $6/x = 2$.

For $x \neq 0$, multiply by x: $6 = 2x$, so $x = 3$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Exponential growth and decay — Higher extension

3.1 Find the y-intercept of $y = 3^x$.

(0,1).

3.2 Evaluate 4×2^x at $x = 3$.

$4 \times 8 = 32$.

3.3 Does $y = (1/2)^x$ grow or decay as x increases?

It decays, because the multiplier is between 0 and 1.

3.4 Write a model for an initial 200 growing by 5% each whole time step.

$P = 200(1.05)^t$, with non-negative integer t for the stated step model.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Find $y = x^3 + 1$ at $x = -2$. [2 marks]

$-8 + 1 = -7$. [1 for cube; 1 for result.]

R2. Find the root of $y = x^3 + 27$. [2 marks]

$x^3 = -27$, so $x = -3$. [1 for isolation; 1 for root.]

R3. For $y = 12/x$, find y at $x = -6$ and state the excluded input. [2 marks]

$y = -2$; $x \neq 0$. [1 each.]

R4. Explain why $y = 1/x$ has no x-intercept. [2 marks]

$1/x$ cannot be zero for any finite non-zero x. Multiplying $1/x = 0$ by an allowed x would give $1 = 0$. [1 for claim; 1 for justification.]

R5. Higher: evaluate $300(1.1)^2$. [2 marks]

$300 \times 1.21 = 363$. [1 for square; 1 for result.]

R6. Higher: compare growth of $y = 3x$ and $y = 3^x$ at $x = 0,1,2$. [3 marks]

Linear values: 0,3,6. Exponential values: 1,3,9. Linear adds 3; exponential multiplies by 3 per unit step. [1 for each list; 1 for interpretation.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 24

24

Real-life and conversion graphs

Shared GCSE core

Graphs communicate relationships between measured quantities. Read scales and units, interpret gradients in context, and distinguish a valid model from an unjustified extrapolation. Include conversion, pricing and distance–time examples.

rate = vertical change / time change

Always read axis labels, scales and units.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Read coordinates, calculate rates and recognise straight lines.

Your route through this notebook

1. Reading scales, interpolating and converting
2. Pricing graphs and comparing models
3. Distance–time graphs and journeys

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Reading scales, interpolating and converting

A graph’s axes identify what is measured and in which units. Read the scale from labelled ticks rather than counting squares as units. Interpolation estimates within the range of known data; extrapolation extends beyond it and depends more strongly on assumptions about the model.

A direct conversion graph often passes through the origin and has constant gradient, such as a fixed exchange-rate model. Other conversions include an offset: Celsius and Fahrenheit do not have a zero-to-zero correspondence. Reverse a conversion by entering on the other axis or rearranging the formula.

A reliable method

1. Identify both quantities, their units and their scales.

2. Read across and up carefully, using the correct input axis.

3. State whether an estimate is interpolation or extrapolation.

Worked example 1

A fixed-rate graph uses D = 1.25P, dollars from pounds. Convert £48.

STEP-BY-STEP SOLUTION

D = 1.25 × 48 = 60 dollars. The model assumes this rate and no fees.

Worked example 2

Use F = 1.8C + 32 to convert 20°C, then find C when F = 68.

STEP-BY-STEP SOLUTION

20°C gives 68°F. Reversing: C = (68 − 32)/1.8 = 20°C.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A conversion line need not pass through the origin. Check the rule: F = 1.8C + 32 has a 32°F intercept even though its gradient is constant.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 At E = 1.2P, convert £35 to euros.

1.2 At E = 1.2P, convert 72 euros to pounds.

1.3 Use F = 1.8C + 32 for C = −10.

1.4 A graph is based on measurements from 2 to 10 minutes. Classify a reading at 6 minutes and a prediction at 14 minutes.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Pricing graphs and comparing models

A fixed fee plus a cost per unit produces a straight line. The intercept is the fixed fee and the gradient is the rate. Two pricing graphs intersect where their costs are equal. On either side, compare their heights to see which is cheaper for the relevant input.

The input domain matters. A phone tariff may round usage to whole units, producing a step graph rather than a perfectly continuous line. A line crossing at a fractional number of tickets still needs interpreting using valid whole-ticket counts. State assumptions such as no minimum charge, no discounts and proportional billing.

A reliable method

1. Write each cost rule with its fixed fee and variable rate.

2. Set costs equal to find the break-even input.

3. Compare costs on either side within the valid domain.

Worked example 1

Plan A costs 10 + 3h pounds and Plan B costs 4 + 5h. Find equal cost.

STEP-BY-STEP SOLUTION

10 + 3h = 4 + 5h gives h = 3. Both cost £19. Below 3 hours B is cheaper; above 3 hours A is cheaper.

Worked example 2

Interpret C = 8 + 2.5d for a delivery.

STEP-BY-STEP SOLUTION

The fixed charge is £8; the gradient is £2.50 per distance unit d. If d is kilometres, the gradient is £2.50/km.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A steeper cost line can still be cheaper over a short range if its fixed fee is lower. Compare the full expressions, not only gradients.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Find cost $C = 6 + 4h$ at $h = 5$.

2.2 Find equal cost for $C = 12 + 2h$ and $C = 4 + 4h$.

2.3 Which of those plans is cheaper at $h = 6$?

2.4 Interpret a cost graph with intercept 15 and gradient 7.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Distance–time graphs and journeys

The gradient of a distance–time graph represents speed when distance travelled is plotted. A horizontal section means distance is unchanged, so the traveller is stationary. A steeper straight section means greater constant speed. Calculate speed using the time and distance changes over the section, not necessarily the coordinates measured from the origin.

A graph labelled distance from home can fall when someone returns towards home; distance travelled cannot decrease. Read the axis label before interpreting direction. Average speed over a whole journey is total distance travelled divided by total elapsed time, including stops if they are part of the interval.

A reliable method

1. Identify whether the vertical axis is distance travelled or distance from a location.

2. Calculate section gradients using differences.

3. For a whole trip, include every travelled section and the relevant elapsed time.

Worked example 1

A traveller covers 6 km in 20 minutes, then rests for 10 minutes. Find moving speed and average speed over 30 minutes.

STEP-BY-STEP SOLUTION

Moving speed = 6/(1/3) = 18 km/h. Whole-interval average = 6/(1/2) = 12 km/h.

Worked example 2

Distance from home falls from 10 km to 4 km over 15 minutes along the same road. Interpret.

STEP-BY-STEP SOLUTION

The traveller moves 6 km towards home. Speed is 6/0.25 = 24 km/h. The negative graph gradient describes decreasing distance from home, not negative speed.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A flat distance–time graph does not mean constant non-zero speed; it means no change in distance. Constant speed corresponds to a straight sloping section.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 A distance–time section rises 8 km in 0.5 hours. Find speed.

3.2 What does a horizontal section mean?

3.3 A cyclist travels 12 km out and 12 km back in 2 hours total. Find average speed.

3.4 A traveller is 9 km from home at 10:00 and 3 km away at 10:20 on a direct return. Find speed.

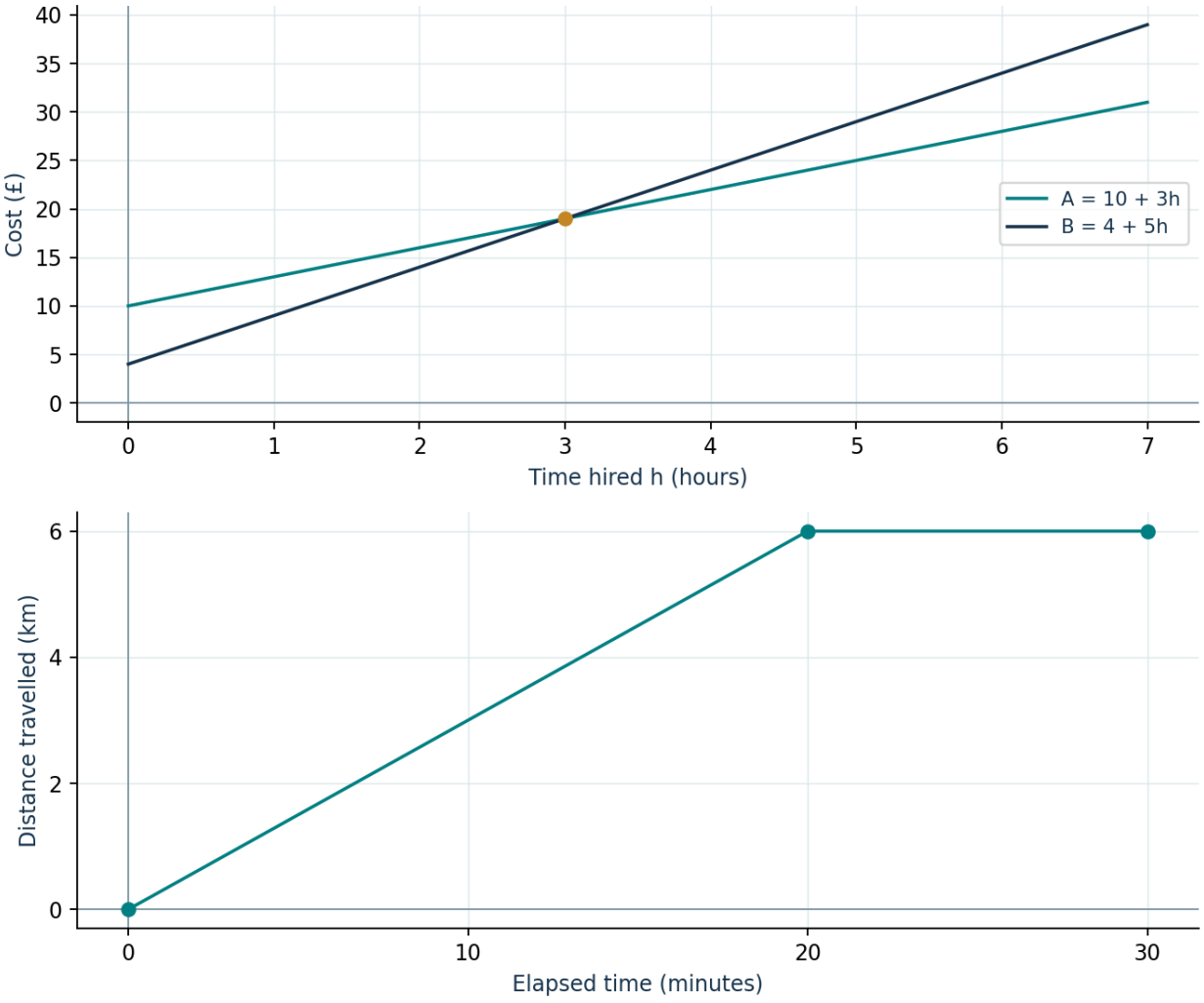
Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

Read quantities and units before interpreting a graph



WHAT THE FIGURE SHOWS

Top: both plans cost £19 at 3 hours; compare the lines on either side. Bottom: the traveller covers 6 km in 20 minutes, then rests for 10 minutes. Moving speed is 18 km/h, while the average over the whole 30 minutes is 12 km/h.

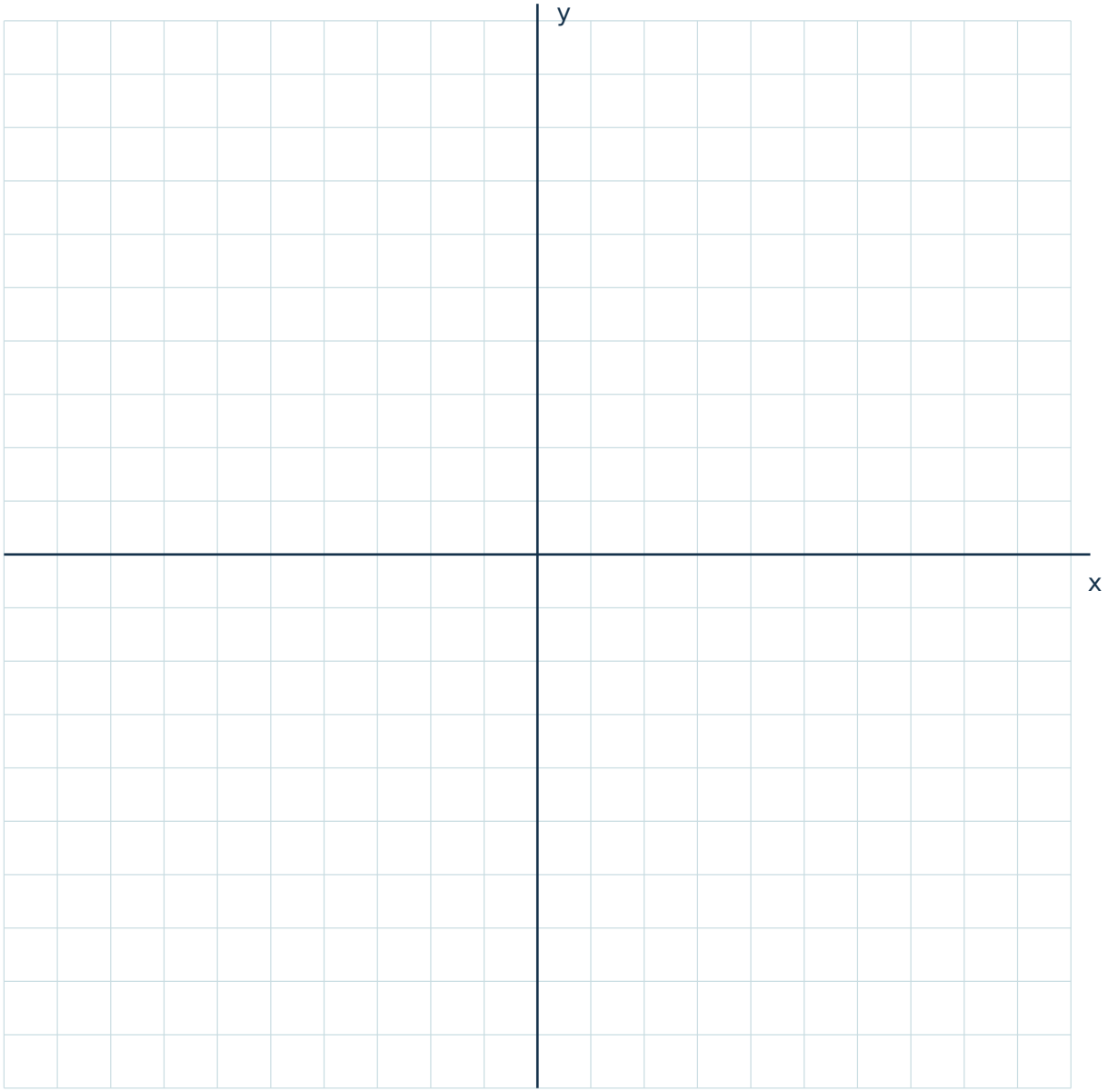
Explain how the labelled features follow from the equations. Check a point by substitution or recalculate a gradient or area before relying on the picture.

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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

Choose one graph from the preceding visual explainer and reconstruct it on this grid. Choose and label a suitable scale on each axis; one square need not represent one unit. Include the displayed domain, key points and units. For a real-life graph, relabel x and y with the measured quantities.



SELF-CHECK

Compare with the preceding plotted figure and its explanation. Check at least three points, any intercepts and the direction or shape of the graph. Explain how the equation or context produces those features.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Use D = 1.3P to convert £60. [2 marks]

R2. Use F = 1.8C + 32 to find C when F = 50. [3 marks]

R3. Find equal cost for A = 18 + 2h and B = 6 + 5h. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Interpret the gradient and intercept in C = 9 + 4n. [2 marks]

R5. A 15 km trip takes 45 minutes including a stop. Find average speed. [3 marks]

R6. A distance-from-home graph drops from 18 km to 6 km in 30 minutes along a direct road. Describe direction and speed. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Reading scales, interpolating and converting

1.1 At E = 1.2P, convert £35 to euros.

E = 42 euros under the fixed-rate model.

1.2 At E = 1.2P, convert 72 euros to pounds.

P = 72/1.2 = £60.

1.3 Use F = 1.8C + 32 for C = −10.

F = −18 + 32 = 14°F.

1.4 A graph is based on measurements from 2 to 10 minutes. Classify a reading at 6 minutes and a prediction at 14 minutes.

6 minutes is interpolation; 14 minutes is extrapolation.

Lesson 2 • Pricing graphs and comparing models

2.1 Find cost C = 6 + 4h at h = 5.

£26.

2.2 Find equal cost for C = 12 + 2h and C = 4 + 4h.

8 = 2h, so h = 4 and cost £20.

2.3 Which of those plans is cheaper at h = 6?

12 + 12 = £24 versus 4 + 24 = £28. The first is cheaper.

2.4 Interpret a cost graph with intercept 15 and gradient 7.

A fixed cost of 15 money units plus 7 per input unit.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Distance–time graphs and journeys

3.1 A distance–time section rises 8 km in 0.5 hours. Find speed.

16 km/h.

3.2 What does a horizontal section mean?

The distance is unchanged; the object is stationary over that interval.

3.3 A cyclist travels 12 km out and 12 km back in 2 hours total. Find average speed.

Total distance 24 km; average speed 12 km/h.

3.4 A traveller is 9 km from home at 10:00 and 3 km away at 10:20 on a direct return. Find speed.

6 km in 1/3 hour gives 18 km/h.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Use D = 1.3P to convert £60. [2 marks]

D = 78 dollars. [1 for multiplication; 1 for result.]

R2. Use F = 1.8C + 32 to find C when F = 50. [3 marks]

C = (50 − 32)/1.8 = 10°C. [1 for subtracting 32; 1 for division; 1 for answer/unit.]

R3. Find equal cost for A = 18 + 2h and B = 6 + 5h. [3 marks]

12 = 3h gives h = 4; both cost £26. [1 for equation; 1 for h; 1 for cost.]

R4. Interpret the gradient and intercept in C = 9 + 4n. [2 marks]

Fixed cost 9; cost per unit n is 4. [1 each.]

R5. A 15 km trip takes 45 minutes including a stop. Find average speed. [3 marks]

45 minutes = 0.75 h; 15/0.75 = 20 km/h. [1 for conversion; 1 for division; 1 for result/unit.]

R6. A distance-from-home graph drops from 18 km to 6 km in 30 minutes along a direct road. Describe direction and speed. [3 marks]

Towards home; 12 km in 0.5 h gives 24 km/h. [1 for direction; 1 for changes; 1 for speed.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 25

25

Velocity–time graphs and rates of change

Higher-level GCSE material

Graph gradients describe rates of change, while areas under velocity graphs describe displacement. Learn exact calculations for straight sections and numerical estimates for curves, keeping speed, velocity, distance and displacement distinct.

gradient → acceleration

Signed area under velocity gives displacement.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Calculate gradients and areas of triangles and trapezia. Read time and velocity units.

Your route through this notebook

1. Gradient as acceleration
2. Area, displacement and distance
3. Curved graphs, tangents and numerical area

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Gradient as acceleration

Velocity includes direction, unlike speed. On a velocity–time graph, gradient is acceleration: change in velocity divided by elapsed time. With velocity in m/s and time in seconds, acceleration is in m/s². A straight sloping section means constant acceleration; a horizontal section means constant velocity.

A negative gradient does not always mean slowing down. If velocity is already negative and becomes more negative, speed increases in the negative direction. Compare the magnitude of velocity with zero to decide whether the object speeds up. A horizontal velocity graph may represent motion, whereas a horizontal distance graph represents rest.

A reliable method

1. Read the velocity change and elapsed time for the chosen section.

2. Divide change in velocity by change in time.

3. Use the sign and velocity values to describe the motion.

Worked example 1

Velocity rises from 4 m/s to 16 m/s in 6 s. Find acceleration.

STEP-BY-STEP SOLUTION

$a = (16 - 4)/6 = 2$ m/s².

Worked example 2

Velocity changes from −2 m/s to −10 m/s in 4 s. Interpret.

STEP-BY-STEP SOLUTION

Acceleration $= (-10 - (-2))/4 = -2$ m/s². Speed grows from 2 to 10 m/s, so it speeds up in the negative direction.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Deceleration is often used for acceleration opposing the motion. A negative acceleration is not sufficient on its own: the sign of the velocity also matters.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Velocity rises from 0 to 18 m/s in 9 s. Find acceleration.

1.2 Velocity falls from 20 to 8 m/s in 4 s. Find acceleration.

1.3 Describe a horizontal graph at v = 5 m/s.

1.4 An object has v = −6 m/s and acceleration +2 m/s². Initially does it speed up or slow down?

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Area, displacement and distance

For a constant velocity, displacement equals velocity multiplied by time, which is the rectangular area under the graph. This extends to changing velocity: signed area between a velocity–time graph and the time axis gives displacement. Straight sections create rectangles, triangles and trapezia whose areas can be found exactly.

Area below the time axis is negative for displacement. Total distance uses the absolute area of each section, so add their magnitudes. If a straight segment crosses the axis, split it at the zero-velocity time before calculating distance. Units provide a check: (m/s) × s = m.

A reliable method

1. Split the region into simple shapes at changes and zero crossings.

2. Use signed area for displacement.

3. For distance, add absolute areas.

Worked example 1

Velocity rises from 0 to 12 m/s in 4 s, then stays at 12 for 3 s. Find displacement.

STEP-BY-STEP SOLUTION

Triangle: 1/2 × 4 × 12 = 24 m. Rectangle: 3 × 12 = 36 m. Total = 60 m.

Worked example 2

Velocity is +4 m/s for 3 s, then −2 m/s for 5 s. Find displacement and distance.

STEP-BY-STEP SOLUTION

Signed areas are +12 m and −10 m. Displacement = 2 m; distance = 12 + 10 = 22 m.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A negative displacement does not mean a negative distance. It describes the net change of position in the chosen coordinate direction.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Find displacement at constant 7 m/s for 8 s.

2.2 Velocity rises linearly from 4 to 10 m/s in 6 s. Find displacement.

2.3 Velocity falls linearly from 8 to 0 m/s in 5 s. Find distance.

2.4 Velocity is −3 m/s for 4 s and +5 m/s for 2 s. Find displacement and distance.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Curved graphs, tangents and numerical area

For a curved graph, the gradient changes. A chord between two points gives an average rate of change. A tangent at one point gives an estimate of instantaneous rate. Draw a tangent that follows the local direction, then use two well-separated points on that tangent to calculate its gradient.

Areas under curves can be estimated by rectangles or trapezia. Trapezia join adjacent sampled heights with straight segments. For equal time intervals h, add each trapezium area $h(v_1 + v_2)/2$. Whether an estimate is high or low depends on the curve shape and the chosen method; it is not always an underestimate.

A reliable method

1. Use a chord for an average rate and a tangent for an instantaneous rate.

2. Read two points on the drawn tangent, not arbitrary points on the curve.

3. For numerical area, include every interval once and retain units.

Worked example 1

A tangent to a velocity curve passes through (2,3) and (6,11). Estimate acceleration.

STEP-BY-STEP SOLUTION

Gradient = $(11 - 3)/(6 - 2) = 2$ m/s².

Worked example 2

At t = 0,2,4 s, velocities are 0,6,8 m/s. Estimate displacement using trapezia.

STEP-BY-STEP SOLUTION

First area = $2(0 + 6)/2 = 6$ m. Second = $2(6 + 8)/2 = 14$ m. Total estimate = 20 m.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A tangent is a local straight-line approximation, not necessarily a line that touches the entire graph only once. Your gradient accuracy depends on a careful drawing and readable scale.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 A tangent passes through (1,4) and (5,14). Find its gradient.

3.2 Velocities at 0,3,6 s are 2,5,8 m/s. Find the trapezium estimate.

3.3 How does a chord differ from a tangent in rate interpretation?

3.4 For a curve bowing below its straight chord, is the chord-based trapezium area an overestimate?

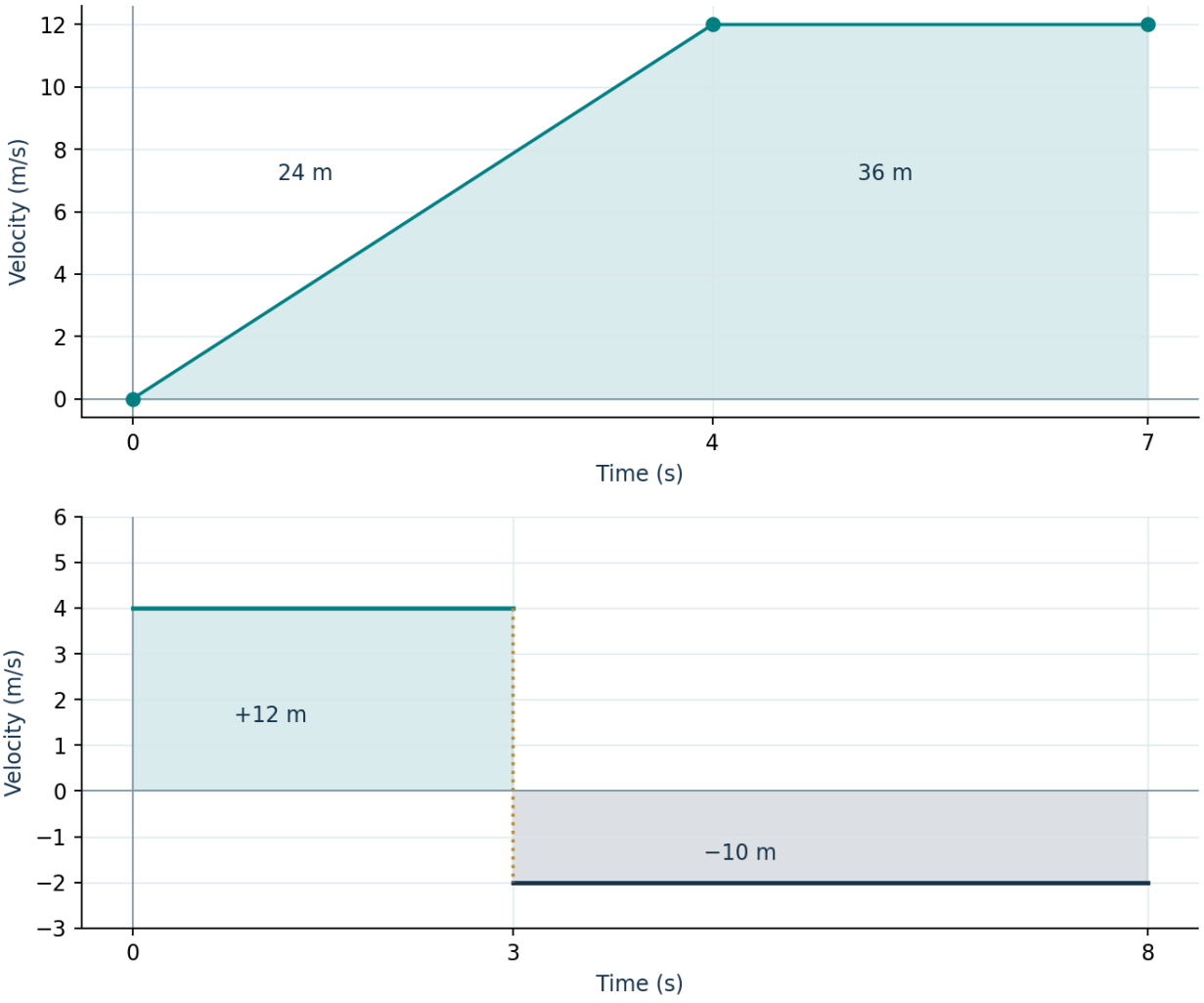
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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

Gradient is acceleration; signed area is displacement



WHAT THE FIGURE SHOWS

Top: acceleration for the first 4 seconds is 12/4 = 3 m/s² and total displacement is 24 + 36 = 60 m. Bottom: an idealised instantaneous velocity change at 3 s separates positive and negative motion. Displacement is 12 − 10 = 2 m; distance is 12 + 10 = 22 m. The dotted jump marks the idealisation, not a finite acceleration segment.

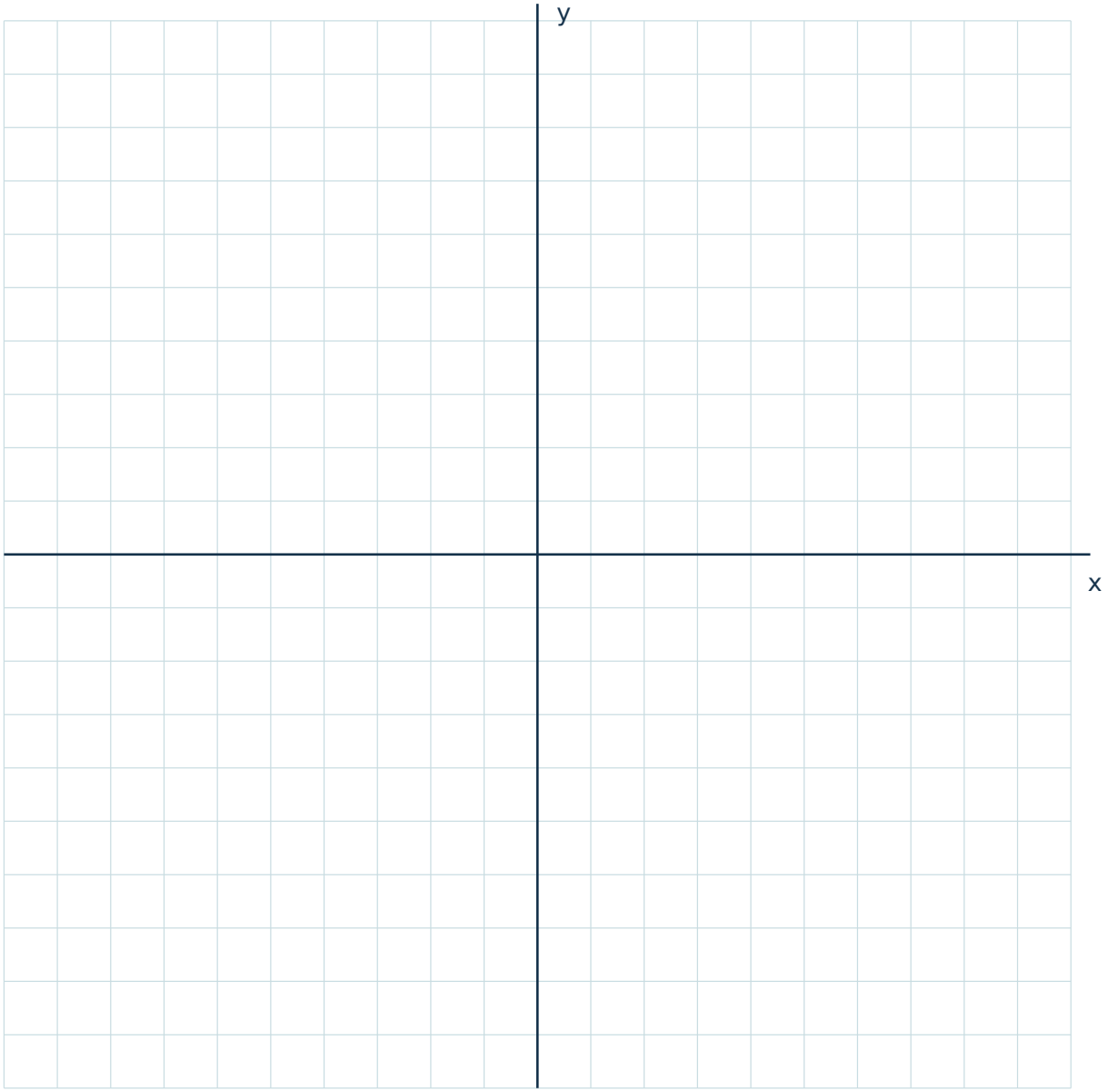
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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

Choose one graph from the preceding visual explainer and reconstruct it on this grid. Choose and label a suitable scale on each axis; one square need not represent one unit. Include the displayed domain, key points and units. For a real-life graph, relabel x and y with the measured quantities.



SELF-CHECK

Compare with the preceding plotted figure and its explanation. Check at least three points, any intercepts and the direction or shape of the graph. Explain how the equation or context produces those features.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Velocity rises from 3 to 15 m/s in 4 s. Find acceleration. [2 marks]

R2. Velocity is 6 m/s for 7 s. Find displacement. [2 marks]

R3. Velocity rises linearly from 2 to 14 m/s in 5 s. Find displacement. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Velocity is +5 m/s for 4 s and −4 m/s for 3 s. Find displacement and distance. [3 marks]

R5. A tangent through (2,5) and (8,17) is drawn on a velocity–time graph. Estimate acceleration. [2 marks]

R6. Velocities at times 0,2,4,6 s are 0,4,7,8 m/s. Use trapezia for displacement. [4 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Gradient as acceleration

1.1 Velocity rises from 0 to 18 m/s in 9 s. Find acceleration.

2 m/s².

1.2 Velocity falls from 20 to 8 m/s in 4 s. Find acceleration.

$(8 - 20)/4 = -3$ m/s².

1.3 Describe a horizontal graph at $v = 5$ m/s.

Constant positive velocity of 5 m/s; acceleration is zero.

1.4 An object has $v = -6$ m/s and acceleration +2 m/s². Initially does it speed up or slow down?

It slows down initially because the acceleration moves its negative velocity towards zero.

Lesson 2 • Area, displacement and distance

2.1 Find displacement at constant 7 m/s for 8 s.

$7 \times 8 = 56$ m.

2.2 Velocity rises linearly from 4 to 10 m/s in 6 s. Find displacement.

Trapezium area $= (4 + 10)/2 \times 6 = 42$ m.

2.3 Velocity falls linearly from 8 to 0 m/s in 5 s. Find distance.

Triangle area $= 1/2 \times 5 \times 8 = 20$ m.

2.4 Velocity is −3 m/s for 4 s and +5 m/s for 2 s. Find displacement and distance.

Displacement $-12 + 10 = -2$ m. Distance $12 + 10 = 22$ m.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Curved graphs, tangents and numerical area

3.1 A tangent passes through (1,4) and (5,14). Find its gradient.

$(14 - 4)/(5 - 1) = 2.5$ in the graph's rate units.

3.2 Velocities at 0,3,6 s are 2,5,8 m/s. Find the trapezium estimate.

$3(2 + 5)/2 + 3(5 + 8)/2 = 10.5 + 19.5 = 30$ m.

3.3 How does a chord differ from a tangent in rate interpretation?

A chord gives average change across an interval; a tangent gives instantaneous change at a point.

3.4 For a curve bowing below its straight chord, is the chord-based trapezium area an overestimate?

Yes, provided the comparison is over the same interval: the straight top lies above the curve.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Velocity rises from 3 to 15 m/s in 4 s. Find acceleration. [2 marks]

$(15 - 3)/4 = 3$ m/s². [1 for changes; 1 for result/unit.]

R2. Velocity is 6 m/s for 7 s. Find displacement. [2 marks]

$6 \times 7 = 42$ m. [1 for area/model; 1 for result.]

R3. Velocity rises linearly from 2 to 14 m/s in 5 s. Find displacement. [3 marks]

$(2 + 14)/2 \times 5 = 40$ m. [1 for trapezium; 1 for substitution; 1 for result.]

R4. Velocity is +5 m/s for 4 s and −4 m/s for 3 s. Find displacement and distance. [3 marks]

Signed areas +20 and −12: displacement 8 m; distance 32 m. [1 for areas; 1 for each total.]

R5. A tangent through (2,5) and (8,17) is drawn on a velocity–time graph. Estimate acceleration. [2 marks]

$12/6 = 2$ m/s². [1 for gradient method; 1 for answer.]

R6. Velocities at times 0,2,4,6 s are 0,4,7,8 m/s. Use trapezia for displacement. [4 marks]

Areas 4,11,15 m; sum 30 m. [1 for each trapezium; 1 for total/unit.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 26

26

Solving quadratic equations

Shared factorisation with Higher methods

Quadratic equations can have two, one or no real solutions. Start with factorisation and the zero-product property, then learn the quadratic formula and completing the square. Choose a method from the equation’s structure and check every solution.

$$ax^2 + bx + c = 0$$

Factorise, complete the square or use the formula.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Factorise quadratics, rearrange equations, simplify roots and use a calculator.

Your route through this notebook

1. Factorisation and the zero-product property
2. The quadratic formula — Higher
3. Completing the square and choosing a method — Higher

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Factorisation and the zero-product property

If a product equals zero, at least one factor must be zero. Therefore a factorised quadratic gives linear equations for its roots. First rearrange the original equation so one side is zero; this is essential because a product equal to a non-zero number does not have the same shortcut.

A common factor x must not be cancelled if doing so loses the solution $x = 0$. Factorise and set each factor equal to zero instead. A repeated factor produces a repeated root, not two distinct values. Numerical checks in the original equation can catch a rearrangement or sign error.

A reliable method

1. Rearrange into a quadratic equal to zero.

2. Factorise completely.

3. Set each factor equal to zero and check the resulting values.

Worked example 1

Solve $x^2 - 5x + 6 = 0$.

STEP-BY-STEP SOLUTION

$(x - 2)(x - 3) = 0$. Thus $x = 2$ or $x = 3$.

Worked example 2

Solve $x^2 = 4x$.

STEP-BY-STEP SOLUTION

$x^2 - 4x = x(x - 4) = 0$. Thus $x = 0$ or 4. Dividing immediately by x would lose zero.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

From $(x - 2)(x - 3) = 6$, you cannot conclude $x = 2$ or 3. Those values make the left side zero, not six. Expand and rearrange to zero first.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Solve $x^2 + 7x + 12 = 0$.

1.2 Solve $x^2 - 16 = 0$.

1.3 Solve $3x^2 - 12x = 0$.

1.4 Solve $x^2 - 6x + 9 = 0$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

The quadratic formula — Higher

For $ax^2 + bx + c = 0$ with $a \neq 0$, the quadratic formula is $x = (-b \pm \sqrt{(b^2 - 4ac)})/(2a)$. Read a,b,c with their signs from a correctly rearranged equation. The entire numerator is divided by 2a. Calculator brackets should reproduce this grouping.

The discriminant $D = b^2 - 4ac$ determines the real-root count: positive gives two distinct real roots, zero gives one repeated real root, and negative gives no real roots. Keep exact roots when requested; otherwise calculate both signs and round only at the end.

A reliable method

1. Put the equation in standard form and identify signed coefficients.

2. Calculate the discriminant before evaluating roots.

3. Use both ± branches and the full denominator.

Worked example 1

Solve $2x^2 + 3x - 1 = 0$ exactly.

STEP-BY-STEP SOLUTION

$a = 2$, $b = 3$, $c = -1$. $D = 9 + 8 = 17$. Thus $x = (-3 \pm \sqrt{17})/4$.

Worked example 2

How many real roots has $x^2 + 2x + 5 = 0$?

STEP-BY-STEP SOLUTION

$D = 2^2 - 4 \times 1 \times 5 = -16$. It has no real roots.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

If b = −5, then −b = +5 and $b^2 = 25$. Enter $(-5)^2$ rather than -5^2 when calculating the discriminant.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Find D for $x^2 - 4x + 1 = 0$.

2.2 Solve $x^2 - 4x + 1 = 0$ exactly.

2.3 Find the root count for $4x^2 + 4x + 1 = 0$.

2.4 Solve $3x^2 - x - 2 = 0$ using the formula.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Completing the square and choosing a method — Higher

Completing the square turns a quadratic equation into a squared expression equal to a constant. For $x^2 + bx + c$, write $(x + b/2)^2 + c - (b/2)^2$. Isolate the square, take both square roots and solve. If the square would equal a negative number, there are no real solutions.

Choose an efficient method: factorisation is quick for obvious integer factors; a simple isolated square may need only square roots; the formula is systematic for any real quadratic. When modelling lengths or counts, solve algebraically first and then reject values that fail the contextual restrictions.

A reliable method

1. Create a complete square and isolate it.

2. Take both signs when the right side is positive.

3. Interpret algebraic solutions in the original domain.

Worked example 1

Solve $x^2 + 6x + 1 = 0$ by completing the square.

STEP-BY-STEP SOLUTION

$(x + 3)^2 - 8 = 0$, so $x + 3 = \pm\sqrt{8}$. Therefore $x = -3 \pm 2\sqrt{2}$.

Worked example 2

A rectangle has sides x and $x + 3$ with area 40. Find positive x.

STEP-BY-STEP SOLUTION

$x^2 + 3x - 40 = 0$ gives $(x + 8)(x - 5) = 0$. Values are -8 and 5; only $x = 5$ is a valid length.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

The instruction “solve” needs values of x, not only completed-square form. Finish by isolating x and include every allowed branch.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Solve $(x - 4)^2 = 9$.

3.2 Solve $x^2 - 8x + 7 = 0$ by completing the square.

3.3 Solve $x^2 + 2x - 2 = 0$ exactly.

3.4 A positive length x satisfies $x(x + 2) = 15$. Find x.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Solve $x^2 - 9x + 20 = 0$. [3 marks]

R2. Solve $2x^2 = 10x$. [3 marks]

R3. Higher: solve $x^2 + 3x - 2 = 0$ exactly. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Higher: state the real-root count of $2x^2 + 4x + 7 = 0$. [2 marks]

R5. Higher: solve $x^2 - 4x - 5 = 0$ by completing the square. [3 marks]

R6. A rectangle has sides x and x + 5 and area 84. Find its dimensions. [4 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Factorisation and the zero-product property

1.1 Solve $x^2 + 7x + 12 = 0$.

$(x + 3)(x + 4) = 0$, so $x = -3$ or -4.

1.2 Solve $x^2 - 16 = 0$.

$(x - 4)(x + 4) = 0$, so $x = \pm 4$.

1.3 Solve $3x^2 - 12x = 0$.

$3x(x - 4) = 0$, so $x = 0$ or 4.

1.4 Solve $x^2 - 6x + 9 = 0$.

$(x - 3)^2 = 0$ gives the repeated root $x = 3$.

Lesson 2 • The quadratic formula — Higher

2.1 Find D for $x^2 - 4x + 1 = 0$.

$16 - 4 = 12$.

2.2 Solve $x^2 - 4x + 1 = 0$ exactly.

$x = (4 \pm \sqrt{12})/2 = 2 \pm \sqrt{3}$.

2.3 Find the root count for $4x^2 + 4x + 1 = 0$.

$D = 16 - 16 = 0$: one repeated real root.

2.4 Solve $3x^2 - x - 2 = 0$ using the formula.

$D = 1 + 24 = 25$. $x = (1 \pm 5)/6$, giving 1 or $-2/3$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Completing the square and choosing a method — Higher

3.1 Solve $(x - 4)^2 = 9$.

$x - 4 = \pm 3$, so $x = 1$ or 7.

3.2 Solve $x^2 - 8x + 7 = 0$ by completing the square.

$(x - 4)^2 = 9$, so $x = 1$ or 7.

3.3 Solve $x^2 + 2x - 2 = 0$ exactly.

$(x + 1)^2 = 3$, so $x = -1 \pm \sqrt{3}$.

3.4 A positive length x satisfies $x(x + 2) = 15$. Find x.

$x^2 + 2x - 15 = (x + 5)(x - 3) = 0$. Positive $x = 3$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Solve $x^2 - 9x + 20 = 0$. [3 marks]

$(x - 4)(x - 5) = 0$, so $x = 4$ or 5. [1 for factors; 1 for each root.]

R2. Solve $2x^2 = 10x$. [3 marks]

$2x(x - 5) = 0$, so $x = 0$ or 5. [1 for factorisation; 1 for each root.]

R3. Higher: solve $x^2 + 3x - 2 = 0$ exactly. [3 marks]

$D = 17$, so $x = (-3 \pm \sqrt{17})/2$. [1 for coefficients/discriminant; 1 for substitution; 1 for both branches.]

R4. Higher: state the real-root count of $2x^2 + 4x + 7 = 0$. [2 marks]

$D = 16 - 56 = -40$, so no real roots. [1 for D; 1 for conclusion.]

R5. Higher: solve $x^2 - 4x - 5 = 0$ by completing the square. [3 marks]

$(x - 2)^2 = 9$, so $x = -1$ or 5. [1 for square; 1 for ±; 1 for values.]

R6. A rectangle has sides x and $x + 5$ and area 84. Find its dimensions. [4 marks]

$x^2 + 5x - 84 = (x + 12)(x - 7) = 0$. Reject -12; sides are 7 and 12. [1 for model; 1 for factors; 1 for valid x; 1 for dimensions.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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ALGEBRA & GRAPHS / TOPIC GUIDE 27

27

Algebraic fractions

Higher-level GCSE material

Algebraic fractions use the same arithmetic principles as numerical fractions, with an extra responsibility: preserve restrictions from original denominators. Factorise before cancelling and distinguish factors from terms.

$$(x^2 - 9)/(x - 3) = x + 3, \quad x \neq 3$$

Cancel factors and preserve original restrictions.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Factorise quadratics, calculate with ordinary fractions and solve equations.

Your route through this notebook

1. Simplifying and preserving the domain
2. Four operations with algebraic fractions
3. Equations with variable denominators

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Simplifying and preserving the domain

A fraction simplifies by cancelling common factors of its whole numerator and denominator. Terms joined by addition are not factors unless the entire expression has first been factorised. For example, $(x^2 + 3x)/x = x(x + 3)/x = x + 3$ for $x \neq 0$. The simplified expression still inherits that restriction.

Cancelling can hide a point where the original expression was undefined. In $(x^2 - 9)/(x - 3)$, factorisation gives $x + 3$ for $x \neq 3$. It is not a licence to evaluate the original at $x = 3$. Denominator restrictions should be identified before any cancellation.

A reliable method

1. List values making any original denominator zero.
2. Factorise whole numerators and denominators.
3. Cancel matching non-zero factors and retain restrictions.

Worked example 1

Simplify $(x^2 - 16)/(x - 4)$.

STEP-BY-STEP SOLUTION

Factorise the numerator: $(x - 4)(x + 4)/(x - 4) = x + 4$, with $x \neq 4$.

Worked example 2

Simplify $(6x^2 + 9x)/(3x)$.

STEP-BY-STEP SOLUTION

$3x(2x + 3)/(3x) = 2x + 3$, with $x \neq 0$.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

(x + 5)/x cannot simplify by crossing out x from only one term. It can be written 1 + 5/x for x ≠ 0, but not as 5.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Simplify $12x^2/(4x)$.

1.2 Simplify $(x^2 - 25)/(x + 5)$.

1.3 Simplify $(x^2 + 5x + 6)/(x + 2)$.

1.4 State the restriction for $1/(2x - 6)$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Four operations with algebraic fractions

Addition and subtraction require a common denominator. Multiply each numerator by whatever was used to change its denominator. Keep a subtraction bracket around an entire numerator, then simplify. Multiplication uses a product of numerators over a product of denominators; factorising first can reveal cancellation.

For division, multiply by the reciprocal of the divisor. The divisor itself must be non-zero, so there may be extra exclusions beyond the original denominators. Record restrictions from every stage that defines the original operation, not only from the final simplified result.

A reliable method

1. For sums, choose a common denominator and preserve numerator brackets.

2. For products, factorise and cancel before multiplying where useful.

3. For division, invert the divisor and exclude zeros of that divisor.

Worked example 1

Simplify $2/x + 3/(x + 1)$.

STEP-BY-STEP SOLUTION

Use $x(x + 1)$: $[2(x + 1) + 3x]/[x(x + 1)] = (5x + 2)/[x(x + 1)]$, with $x \neq 0, -1$.

Worked example 2

Simplify $(x^2 - 1)/(x + 2) \div (x - 1)/(x + 2)$.

STEP-BY-STEP SOLUTION

Multiply by $(x + 2)/(x - 1)$ and factor $x^2 - 1$. Result $x + 1$, with $x \neq -2$ from denominators and $x \neq 1$ because the divisor cannot be zero.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

For $1/x - 1/(x + 1)$, the numerator is $(x + 1) - x = 1$. Neglecting the subtraction bracket is a common source of false terms.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Simplify $3/x + 5/x$.

2.2 Simplify $1/x - 1/(x + 2)$.

2.3 Simplify $(2x/3) \times (9/x^2)$.

2.4 Simplify $(x/4) \div (x^2/8)$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Equations with variable denominators

To solve an equation containing algebraic fractions, first state values excluded by the denominators. Multiply every term by a common denominator to form a polynomial equation. Solving that equation may produce values that were invalid in the original, so check all candidates against the restrictions.

A denominator with a variable can change sign, but for an equation multiplying by a non-zero expression preserves equality. Inequalities require additional sign reasoning and are not handled by blindly using the same step. Keep the focus here on equations and verify by substitution.

A reliable method

1. Write all denominator restrictions first.

2. Multiply every term by the common denominator.

3. Solve, reject excluded values and check the original equation.

Worked example 1

Solve $3/x = 2/(x + 1)$.

STEP-BY-STEP SOLUTION

$x \neq 0, -1$. Cross-multiplication gives $3(x + 1) = 2x$, so $x = -3$, which is allowed and checks.

Worked example 2

Solve $1/(x - 1) + 1 = 3/(x - 1)$.

STEP-BY-STEP SOLUTION

$x \neq 1$. Multiply by $x - 1$: $1 + x - 1 = 3$, so $x = 3$. Check: $1/2 + 1 = 3/2$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Clearing a denominator can produce an excluded candidate. From $(x^2 - 1)/(x - 1) = 2$, algebra leads to $x = 1$, but the original is undefined there. Hence the original equation has no solution.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Solve $5/x = 1$.

3.2 Solve $2/(x + 1) = 1$.

3.3 Solve $1/x + 1/2 = 1$.

3.4 Solve $(x^2 - 4)/(x - 2) = 4$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Simplify $(x^2 - 36)/(x - 6)$ and state restrictions. [3 marks]

R2. Simplify $(4x^2 + 8x)/(2x)$. [3 marks]

R3. Simplify $2/x + 1/(x - 1)$. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Simplify $((x + 1)/x) \times (x^2/(x^2 - 1))$. [3 marks]

R5. Solve $4/x = 3/(x + 2)$. [3 marks]

R6. Solve $(x^2 - 9)/(x - 3) = 6$. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Simplifying and preserving the domain

1.1 Simplify $12x^2/(4x)$.

$3x$, with $x \neq 0$.

1.2 Simplify $(x^2 - 25)/(x + 5)$.

$x - 5$, with $x \neq -5$.

1.3 Simplify $(x^2 + 5x + 6)/(x + 2)$.

$(x + 2)(x + 3)/(x + 2) = x + 3$, with $x \neq -2$.

1.4 State the restriction for $1/(2x - 6)$.

$2x - 6 \neq 0$, so $x \neq 3$.

Lesson 2 • Four operations with algebraic fractions

2.1 Simplify $3/x + 5/x$.

$8/x$, $x \neq 0$.

2.2 Simplify $1/x - 1/(x + 2)$.

$2/[x(x + 2)]$, $x \neq 0, -2$.

2.3 Simplify $(2x/3) \times (9/x^2)$.

$6/x$, $x \neq 0$.

2.4 Simplify $(x/4) \div (x^2/8)$.

$(x/4)(8/x^2) = 2/x$, $x \neq 0$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Equations with variable denominators

3.1 Solve $5/x = 1$.

$x \neq 0$; $x = 5$.

3.2 Solve $2/(x + 1) = 1$.

$x \neq -1$; $2 = x + 1$ gives $x = 1$.

3.3 Solve $1/x + 1/2 = 1$.

$x \neq 0$. Multiply by $2x$: $2 + x = 2x$, so $x = 2$.

3.4 Solve $(x^2 - 4)/(x - 2) = 4$.

$x \neq 2$. Simplifies to $x + 2 = 4$, giving excluded $x = 2$. No solution.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Simplify $(x^2 - 36)/(x - 6)$ and state restrictions. [3 marks]

$x + 6$, $x \neq 6$. [1 for factors; 1 for cancellation; 1 for restriction.]

R2. Simplify $(4x^2 + 8x)/(2x)$. [3 marks]

$2x(x + 2)/(x)$ after coefficient simplification, or directly $2(x + 2) = 2x + 4$, $x \neq 0$. [1 for factorisation; 1 for cancellation; 1 for result/restriction.]

R3. Simplify $2/x + 1/(x - 1)$. [3 marks]

$[2(x - 1) + x]/[x(x - 1)] = (3x - 2)/[x(x - 1)]$, $x \neq 0,1$. [1 for denominator; 1 for numerator; 1 for simplified form/restrictions.]

R4. Simplify $((x + 1)/x) \times (x^2/(x^2 - 1))$. [3 marks]

Factor $x^2 - 1 = (x - 1)(x + 1)$. Result $x/(x - 1)$, with $x \neq 0,1,-1$. [1 for factors; 1 for cancellation; 1 for restrictions.]

R5. Solve $4/x = 3/(x + 2)$. [3 marks]

$x \neq 0,-2$; $4x + 8 = 3x$ gives $x = -8$. [1 for restrictions; 1 for equation; 1 for allowed result.]

R6. Solve $(x^2 - 9)/(x - 3) = 6$. [3 marks]

$x \neq 3$. Simplification gives $x + 3 = 6$, so the only candidate is excluded $x = 3$. No solution. [1 for restriction; 1 for candidate; 1 for rejecting it.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 28

28

Functions and graph transformations

Shared function machines with Higher extensions

A function assigns one output to each allowed input. Develop notation, composition and inversion, then connect changes in a function’s rule to movements of its graph. Domain restrictions are kept explicit.

f(g(x)): g acts first

Track inputs, outputs and the order of operations.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

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Before you begin

Substitute values, rearrange formulae and recognise basic graph families.

Your route through this notebook

1. Function notation and domain
2. Composite and inverse functions — Higher
3. Translations and reflections — Higher

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Function notation and domain

The notation f(x) means the output of function f for input x; it does not mean f multiplied by x. In f(x) = 3x − 2, replace every x by the supplied input, including an expression such as a + 1. A function must give a single output for each input in its domain.

The domain is the set of permitted inputs. For a rational expression, denominator zeros must be excluded. For a real square-root function, the quantity under the root must be non-negative. Context may impose tighter restrictions, such as non-negative whole-number ticket counts.

A reliable method

1. Read f(input) as a substitution instruction.

2. Bracket expression inputs before simplifying.

3. Check denominator and square-root restrictions.

Worked example 1

For $f(x) = 2x^2 - 1$, find $f(-3)$.

STEP-BY-STEP SOLUTION

$f(-3) = 2(-3)^2 - 1 = 17$.

Worked example 2

For f(x) = 3x + 4, find f(a − 2).

STEP-BY-STEP SOLUTION

f(a − 2) = 3(a − 2) + 4 = 3a − 2.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

An input-output machine and function notation describe the same idea. The order of steps matters: multiply by 3 then add 4 is not the same as add 4 then multiply by 3.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 For $f(x) = 5x - 2$, find $f(4)$.

1.2 For $f(x) = x^2 + 3$, find $f(-2)$.

1.3 For $f(x) = 1/(x - 5)$, state the excluded input.

1.4 For $f(x) = \sqrt{(x + 2)}$, state the real domain.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Composite and inverse functions — Higher

A composite function applies one function and then another. In fg(x), meaning f(g(x)), g acts first. Substitute the entire output expression of g into f. Reversing the order generally changes the result. This guide explicitly uses fg for composition rather than multiplication.

An inverse reverses a one-to-one function. Write y = f(x), rearrange to express x in terms of y, then relabel the input. The notation $f^{-1}(x)$ means inverse function, not 1/f(x). A function such as x^2 needs a restricted domain to have a single-valued inverse, because positive and negative inputs otherwise share outputs.

A reliable method

1. For f(g(x)), work from the inner function outward.

2. For an inverse, rearrange y = f(x).

3. Check the domain and verify a round trip.

Worked example 1

Let f(x) = 2x + 1 and $g(x) = x^2$. Find f(g(x)) and g(f(x)).

STEP-BY-STEP SOLUTION

$f(g(x)) = 2x^2 + 1$. $g(f(x)) = (2x + 1)^2$. They differ in general.

Worked example 2

Find the inverse of f(x) = 3x − 7.

STEP-BY-STEP SOLUTION

$y = 3x - 7$ gives $x = (y + 7)/3$. Therefore $f^{-1}(x) = (x + 7)/3$.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

For $f(x) = x^2$ on $x \geq 0$, the inverse is $\sqrt{x}$ with input $x \geq 0$. Without the original restriction, the proposed inverse would not recover a negative original input.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 If $f(x) = x + 4$ and $g(x) = 2x$, find $f(g(x))$.

2.2 With those functions, find $g(f(x))$.

2.3 Find the inverse of $f(x) = 5x + 2$.

2.4 Find the inverse of $f(x) = x^2 + 1$ on $x \geq 0$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Translations and reflections — Higher

A vertical translation changes outputs: y = f(x) + a moves the graph up a. A horizontal translation changes inputs: y = f(x − a) moves it right a. The horizontal sign can feel reversed because the old input is reached when the new x is larger by a.

For y = −f(x), every output changes sign, giving a reflection in the x-axis. For y = f(−x), inputs change sign, giving a reflection in the y-axis. Track a known point instead of relying only on a verbal rule. Multiple changes can be understood by writing the new coordinate of a general original point.

A reliable method

1. Identify whether the change is outside or inside f.

2. Track a known point (p,q).

3. Check the new rule by substitution at the transformed point.

Worked example 1

The graph y = f(x) contains (2,5). Where does it go on y = f(x − 3) + 1?

STEP-BY-STEP SOLUTION

Right 3 and up 1: (5,6). At x = 5, the input to f is 2 and the output is 5 + 1.

Worked example 2

Reflect $y = x^2 - 2$ in the x-axis.

STEP-BY-STEP SOLUTION

Negate the entire output: $y = -(x^2 - 2) = -x^2 + 2$.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

y = f(x + 4) moves left 4, not right. For a point originally at input 2, the new x must be −2 so that x + 4 still equals 2.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Describe $y = f(x) - 6$.

3.2 Describe $y = f(x + 2)$.

3.3 A point (−3,4) is reflected in the y-axis. Find its image.

3.4 The vertex of $y = x^2$ is (0,0). Find the vertex of $y = (x - 4)^2 - 3$.

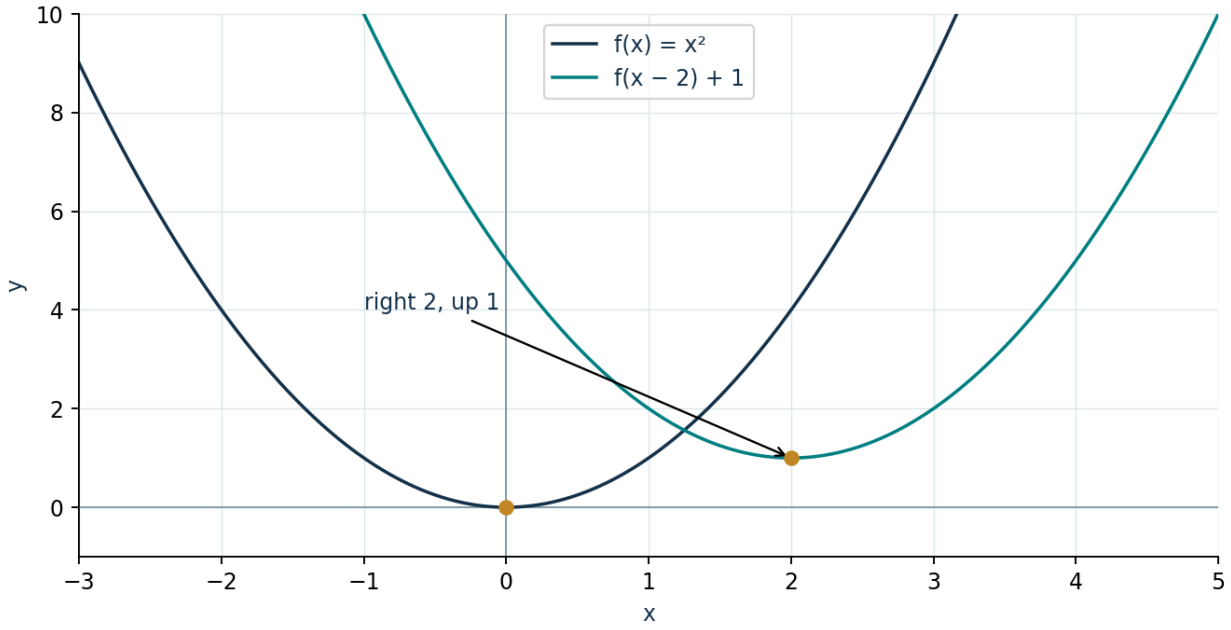
Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

Track points when transforming a graph



WHAT THE FIGURE SHOWS

The original vertex (0,0) moves to (2,1). More generally, a point (p,q) moves to (p + 2,q + 1). The input x − 2 reaches the old input p when x = p + 2, explaining the horizontal sign.

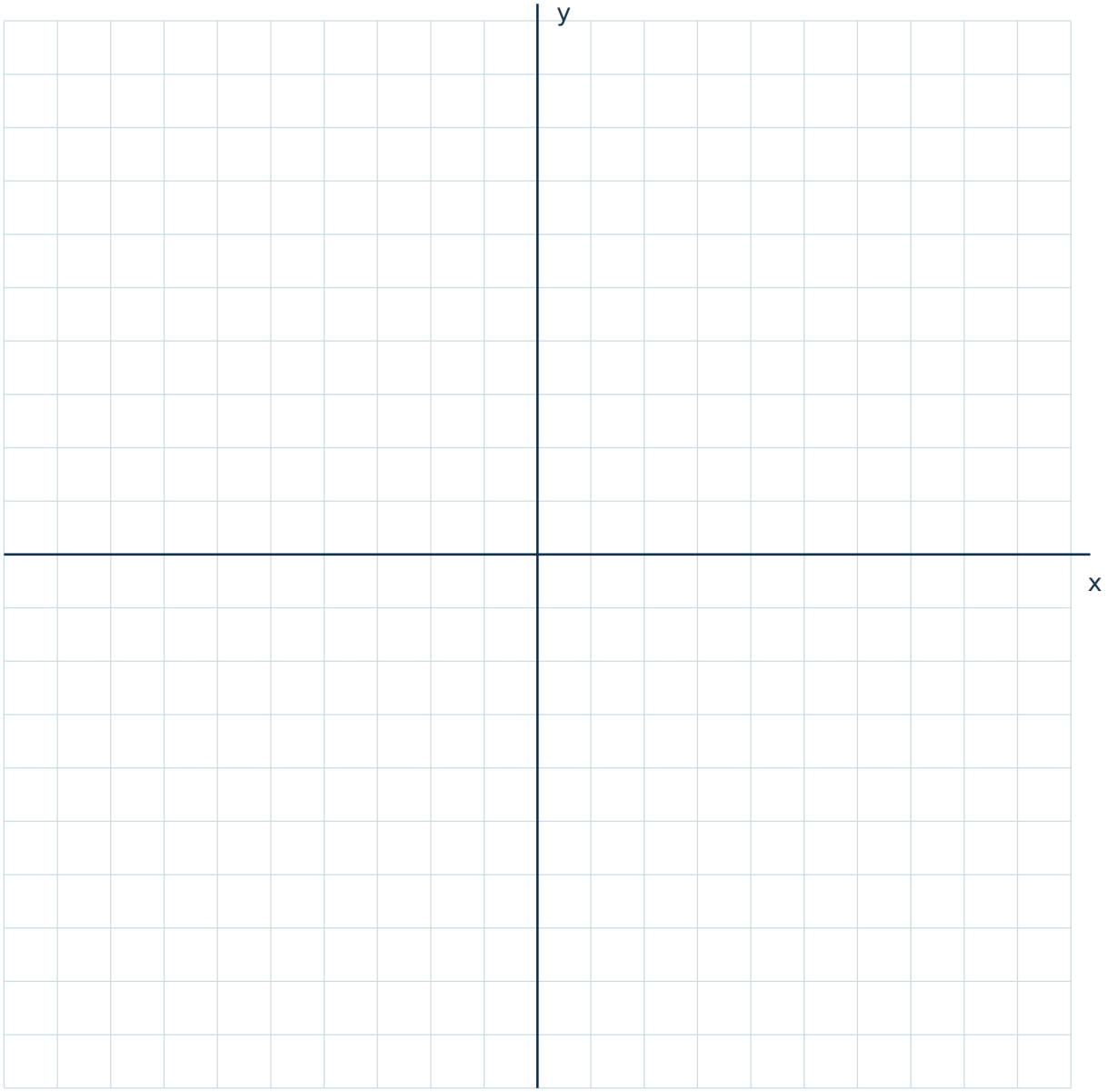
Explain how the labelled features follow from the equations. Check a point by substitution or recalculate a gradient or area before relying on the picture.

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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

Choose one graph from the preceding visual explainer and reconstruct it on this grid. Choose and label a suitable scale on each axis; one square need not represent one unit. Include the displayed domain, key points and units. For a real-life graph, relabel x and y with the measured quantities.



SELF-CHECK

Compare with the preceding plotted figure and its explanation. Check at least three points, any intercepts and the direction or shape of the graph. Explain how the equation or context produces those features.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. For $f(x) = 3x^2 + 2$, find $f(-2)$. [2 marks]

R2. For $f(x) = 2x - 5$, simplify $f(a + 3)$. [2 marks]

R3. Higher: $f(x) = 3x + 1$, $g(x) = x - 2$. Find $f(g(x))$. [2 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Higher: find the inverse of $f(x) = 4x - 9$. [3 marks]

R5. Higher: describe $y = f(x - 5) + 2$. [2 marks]

R6. Higher: a point $(2,-3)$ lies on $y = f(x)$. Give its image on $y = -f(x + 1)$. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Function notation and domain

1.1 For $f(x) = 5x - 2$, find $f(4)$.

18.

1.2 For $f(x) = x^2 + 3$, find $f(-2)$.

7.

1.3 For $f(x) = 1/(x - 5)$, state the excluded input.

$x \neq 5$.

1.4 For $f(x) = \sqrt{(x + 2)}$, state the real domain.

$x \geq -2$.

Lesson 2 • Composite and inverse functions — Higher

2.1 If $f(x) = x + 4$ and $g(x) = 2x$, find $f(g(x))$.

$2x + 4$.

2.2 With those functions, find $g(f(x))$.

$2x + 8$.

2.3 Find the inverse of $f(x) = 5x + 2$.

$f^{-1}(x) = (x - 2)/5$.

2.4 Find the inverse of $f(x) = x^2 + 1$ on $x \geq 0$.

$f^{-1}(x) = \sqrt{(x - 1)}$, with inverse input $x \geq 1$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Translations and reflections — Higher

3.1 Describe $y = f(x) - 6$.

Translation down 6.

3.2 Describe $y = f(x + 2)$.

Translation left 2.

3.3 A point $(-3,4)$ is reflected in the y-axis. Find its image.

$(3,4)$.

3.4 The vertex of $y = x^2$ is $(0,0)$. Find the vertex of $y = (x - 4)^2 - 3$.

$(4,-3)$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. For $f(x) = 3x^2 + 2$, find $f(-2)$. [2 marks]

$3 \times 4 + 2 = 14$. [1 for substitution; 1 for result.]

R2. For $f(x) = 2x - 5$, simplify $f(a + 3)$. [2 marks]

$2(a + 3) - 5 = 2a + 1$. [1 for substitution; 1 for simplification.]

R3. Higher: $f(x) = 3x + 1$, $g(x) = x - 2$. Find $f(g(x))$. [2 marks]

$3(x - 2) + 1 = 3x - 5$. [1 for order; 1 for simplification.]

R4. Higher: find the inverse of $f(x) = 4x - 9$. [3 marks]

$y = 4x - 9$ gives $x = (y + 9)/4$, so $f^{-1}(x) = (x + 9)/4$. [1 for equation; 1 for rearrangement; 1 for notation.]

R5. Higher: describe $y = f(x - 5) + 2$. [2 marks]

Translation right 5 and up 2. [1 each.]

R6. Higher: a point $(2,-3)$ lies on $y = f(x)$. Give its image on $y = -f(x + 1)$. [3 marks]

Left 1 and reflect output: (1,3). [1 for x change; 1 for y change; 1 for paired coordinate.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 29

29

Numerical methods and iteration

Higher-level GCSE material

Some equations are easier to solve approximately than exactly. Learn how a sign change brackets a root, how repeated substitution generates estimates and how to decide whether a requested rounded answer is justified.

$$x_{n+1} = g(x_n)$$

Iterate carefully; use a bracket to justify accuracy.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

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Before you begin

Substitute into functions, rearrange equations, use roots and retain calculator precision.

Your route through this notebook

1. Locating roots by trial and sign change
2. Repeated substitution and iterative formulae
3. Bisection and proving decimal accuracy

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Locating roots by trial and sign change

For a continuous function, a change of sign between two inputs guarantees at least one root between them. Calculate the function at the two endpoints and state both the sign change and continuity assumption. This does not prove a unique root unless additional reasoning, such as monotonicity, is available.

A sign change across a discontinuity is not a valid root guarantee: 1/x changes sign across zero but has no root and is undefined at zero. Once a valid interval is found, test points inside it to narrow the bracket. Keep the function definition and every calculated value visible.

A reliable method

1. Rearrange the equation as f(x) = 0.

2. Find endpoint values of opposite signs on a continuous interval.

3. Test intermediate values and retain the smaller sign-changing interval.

Worked example 1

Show a root of $x^3 - x - 2 = 0$ lies between 1 and 2.

STEP-BY-STEP SOLUTION

f(1) = −2 and f(2) = 4. The polynomial is continuous, so a sign change guarantees a root in (1,2).

Worked example 2

Narrow that root using x = 1.5 and x = 1.6.

STEP-BY-STEP SOLUTION

f(1.5) = −0.125 and f(1.6) = 0.496. Therefore a root lies in (1.5,1.6).

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Two positive endpoint values do not prove there is no root between them. A curve could cross the axis twice. A sign-change test is sufficient under continuity, but not necessary for every root.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Bracket a root of $x^2 - 5 = 0$ between consecutive positive integers.

1.2 Evaluate $x^3 - 4$ at $x = 1$ and $x = 2$ and interpret.

1.3 Why does the sign change of $1/x$ between -1 and 1 not prove a root?

1.4 For $f(x) = x^2 - 2$, calculate $f(1.4)$ and $f(1.5)$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Repeated substitution and iterative formulae

An iterative formula generates a new estimate from the previous one: $x_{n+1} = g(x_n)$. Start from the supplied x_0 and substitute each result back into the same rule. Keep full calculator precision internally, even if the displayed table is rounded. Otherwise rounding can change later estimates.

A rearranged fixed point satisfies $x = g(x)$, but not every rearrangement converges to it. Some iterations oscillate or diverge. If values settle down numerically, they suggest a solution; a residual check in the original equation and a valid bracket give stronger support. Read subscripts carefully: x_0 is the starting value, not the first computed update.

A reliable method

1. Record the starting value with its subscript.

2. Compute each update from the previous unrounded estimate.

3. Check the apparent limit in the original equation.

Worked example 1

Use $x_{n+1} = (x_n + 6/x_n)/2$ with $x_0 = 2$. Find x_1 and x_2.

STEP-BY-STEP SOLUTION

$x_1 = (2 + 3)/2 = 2.5$. $x_2 = (2.5 + 2.4)/2 = 2.45$. This iteration estimates $\sqrt{6}$.

Worked example 2

Rearrange $x^3 - x - 2 = 0$ into a cube-root iteration.

STEP-BY-STEP SOLUTION

$x^3 = x + 2$, so use x_{n+1} = cube root of $(x_n + 2)$. A suitable start near the positive root produces improving estimates.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A repeated calculator display is evidence of numerical stability at that display precision, not by itself a rigorous proof of the correctly rounded mathematical root.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Use $x_{n+1} = (x_n + 4)/2$, $x_0 = 0$, to find x_1, x_2, x_3.

2.2 Find the fixed point of $x_{n+1} = (x_n + 4)/2$.

2.3 Use $x_{n+1} = \sqrt{(x_n + 2)}$, $x_0 = 1$, to find x_1 and x_2 to 4 decimal places.

2.4 For $x_{n+1} = 2x_n$, $x_0 = 1$, list the next four values and describe.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Bisection and proving decimal accuracy

Bisection halves a valid sign-changing interval. Test the midpoint, then keep the half whose endpoints still have opposite signs. For a continuous function this retains at least one root. Repeating reduces the interval width predictably, though a uniqueness argument may still be needed for a specific root.

To justify a rounded value, ensure the whole final bracket lies inside that value's rounding interval. For example, an answer 1.52 to 2 decimal places is secure if the root is bracketed between 1.515 and 1.525. Testing 1.52 itself and finding a small residual does not alone establish the rounding.

A reliable method

1. Find the midpoint and calculate its function value.

2. Retain opposite-sign endpoints.

3. Compare final endpoints with rounding boundaries, not just decimal digits.

Worked example 1

Apply one bisection step to $f(x) = x^2 - 2$ on $[1,2]$.

STEP-BY-STEP SOLUTION

Midpoint 1.5 gives $f(1.5) = 0.25$. Since $f(1) = -1$, retain $[1,1.5]$.

Worked example 2

A root is bracketed between 1.521 and 1.522. Give it to 2 decimal places.

STEP-BY-STEP SOLUTION

Both endpoints and every point between round to 1.52, so the root is 1.52 to 2 decimal places.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

An interval from 1.514 to 1.516 is narrow but crosses the 1.515 rounding boundary. It does not establish a single 2-decimal-place answer. Width alone is not enough.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Bisect [2,3] once for $f(x) = x^2 - 5$.

3.2 A root lies in (2.236,2.237). Give it to 2 decimal places.

3.3 A root lies in (4.444,4.446). Is 4.44 guaranteed to 2 decimal places?

3.4 After four bisections, what is the width of an initial interval of width 2?

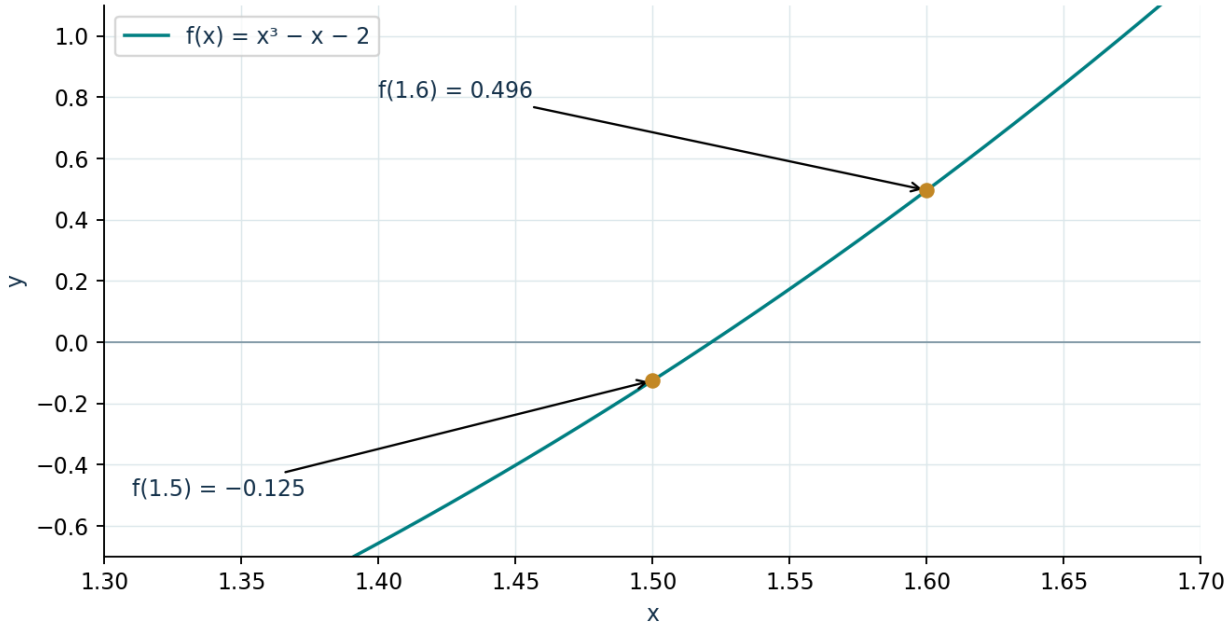
Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

A continuous sign change brackets a root



WHAT THE FIGURE SHOWS

The polynomial is continuous and changes sign between 1.5 and 1.6. A root lies inside that interval. To establish a rounded answer, narrow the bracket until every possible value within it rounds identically; a graph alone usually cannot justify many decimal places.

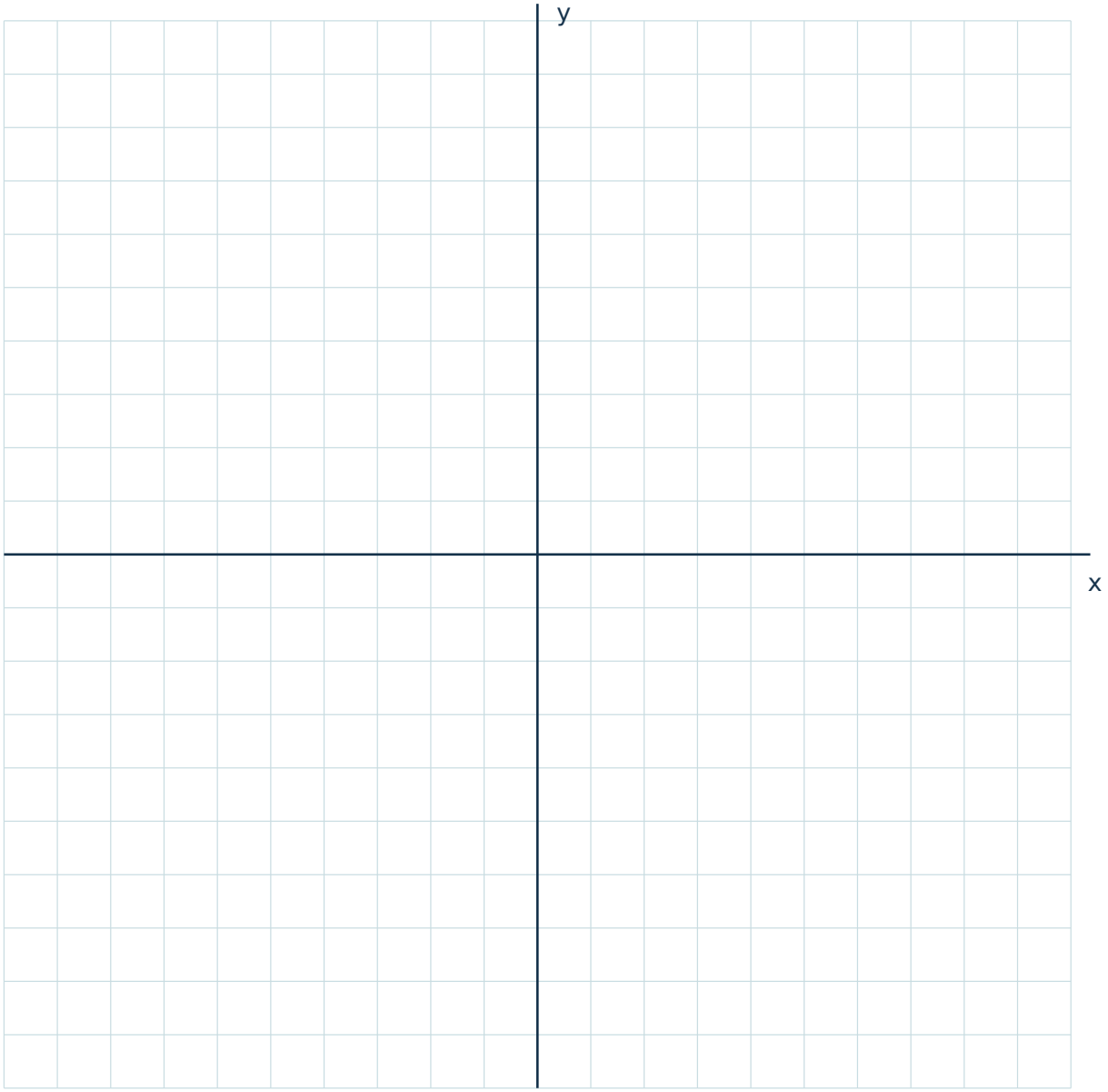
Explain how the labelled features follow from the equations. Check a point by substitution or recalculate a gradient or area before relying on the picture.

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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

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SELF-CHECK

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Show a root of $x^3 - 5 = 0$ lies between 1 and 2. [3 marks]

R2. For $f(x) = x^3 - 5$, evaluate at 1.7 and 1.8 and narrow the interval. [3 marks]

R3. Use $x_{n+1} = (x_n + 8/x_n)/2$ with $x_0 = 2$. Find x_1, x_2. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Find the fixed point of $x_{n+1} = (x_n + 10)/3$. [2 marks]

R5. A root lies between 1.709 and 1.710. State it to 2 decimal places and justify. [2 marks]

R6. Explain why $x_{n+1} = 3x_n$ with $x_0 = 1$ does not converge. [2 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Locating roots by trial and sign change

1.1 Bracket a root of $x^2 - 5 = 0$ between consecutive positive integers.

$f(2) = -1$ and $f(3) = 4$, so a root lies between 2 and 3.

1.2 Evaluate $x^3 - 4$ at $x = 1$ and $x = 2$ and interpret.

-3 and 4; continuity gives a root between 1 and 2.

1.3 Why does the sign change of $1/x$ between -1 and 1 not prove a root?

The function is not continuous on the interval: it is undefined at 0.

1.4 For $f(x) = x^2 - 2$, calculate $f(1.4)$ and $f(1.5)$.

-0.04 and 0.25, so the positive root lies between 1.4 and 1.5.

Lesson 2 • Repeated substitution and iterative formulae

2.1 Use $x_{n+1} = (x_n + 4)/2$, $x_0 = 0$, to find x_1, x_2, x_3.

2,3,3.5.

2.2 Find the fixed point of $x_{n+1} = (x_n + 4)/2$.

Set $x = (x + 4)/2$: $x = 4$.

2.3 Use $x_{n+1} = \sqrt{(x_n + 2)}$, $x_0 = 1$, to find x_1 and x_2 to 4 decimal places.

$x_1 = \sqrt{3} \approx 1.7321$; $x_2 = \sqrt{(\sqrt{3} + 2)} \approx 1.9319$. Retain unrounded x_1 internally.

2.4 For $x_{n+1} = 2x_n$, $x_0 = 1$, list the next four values and describe.

2,4,8,16. The iteration diverges rather than settling to a finite value.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Bisection and proving decimal accuracy

3.1 Bisect [2,3] once for $f(x) = x^2 - 5$.

Midpoint 2.5 gives $1.25 > 0$; $f(2) < 0$. Retain [2,2.5].

3.2 A root lies in (2.236,2.237). Give it to 2 decimal places.

2.24.

3.3 A root lies in (4.444,4.446). Is 4.44 guaranteed to 2 decimal places?

No: the interval crosses 4.445, so possible roots round to 4.44 or 4.45.

3.4 After four bisections, what is the width of an initial interval of width 2?

$2/2^4 = 0.125$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Show a root of $x^3 - 5 = 0$ lies between 1 and 2. [3 marks]

$f(1) = -4$, $f(2) = 3$; the continuous polynomial changes sign. [1 for each value; 1 for continuity/conclusion.]

R2. For $f(x) = x^3 - 5$, evaluate at 1.7 and 1.8 and narrow the interval. [3 marks]

$f(1.7) = -0.087$; $f(1.8) = 0.832$. A root lies in (1.7,1.8). [1 for each value; 1 for interval.]

R3. Use $x_{n+1} = (x_n + 8/x_n)/2$ with $x_0 = 2$. Find x_1,x_2. [3 marks]

$x_1 = 3$; $x_2 = (3 + 8/3)/2 = 17/6 \approx 2.8333$. [1 for first update; 1 for second substitution; 1 for result.]

R4. Find the fixed point of $x_{n+1} = (x_n + 10)/3$. [2 marks]

$3x = x + 10$ gives $x = 5$. [1 for fixed-point equation; 1 for result.]

R5. A root lies between 1.709 and 1.710. State it to 2 decimal places and justify. [2 marks]

1.71, because the entire bracket lies in [1.705,1.715). [1 for value; 1 for rounding interval.]

R6. Explain why $x_{n+1} = 3x_n$ with $x_0 = 1$ does not converge. [2 marks]

The values 1,3,9,27,... grow without bound. [1 for example updates; 1 for divergence.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 30

30

Algebraic reasoning and proof

Shared reasoning with Higher extensions

A proof explains why a statement holds for every permitted case. Use algebra to represent general integers, compare equivalent expressions and construct logical arguments. Learn when a counterexample is enough and when numerical testing is insufficient.

n(n + 1) is even for integer n

A general argument proves more than examples.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Before you begin

Expand brackets, factorise expressions and understand integers, parity and divisibility.

Your route through this notebook

1. Conjectures, counterexamples and identities
2. Parity, consecutive integers and divisibility
3. Products, squares and structured arguments — Higher

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Conjectures, counterexamples and identities

A conjecture is a statement that may be true but needs justification. Testing examples can suggest a pattern but does not prove a universal claim. A single counterexample disproves a statement asserting “for all” or “always”. To prove an identity, transform one side into the other using valid algebra, or simplify both sides to the same expression.

An equation can hold for only some inputs, whereas an identity holds for every permitted input. Domain restrictions still matter: $x/x = 1$ is an identity only for $x \neq 0$. Show connected reasoning rather than assuming the desired equality at the start.

A reliable method

1. Identify whether the claim is universal or about a specific input.

2. Use a counterexample to disprove an always-claim.

3. For proof, use general variables and valid transformations.

Worked example 1

Disprove “every prime is odd”.

STEP-BY-STEP SOLUTION

The prime 2 is even, so it is a counterexample.

Worked example 2

Prove $3(x + 2) + 2(x - 1) = 5x + 4$.

STEP-BY-STEP SOLUTION

Expand the left side: $3x + 6 + 2x - 2 = 5x + 4$. This matches the right side for every real x.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Checking $x = 1$ and $x = 2$ is not proof that two expressions are identical. Different expressions can agree at selected points and differ elsewhere.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Disprove “every odd number is prime”.

1.2 Show $4(x + 1) - x$ is identical to $3x + 4$.

1.3 Is $x + 2 = 5$ an identity? Explain.

1.4 State the restriction for $x^2/x = x$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Parity, consecutive integers and divisibility

An even integer can be written 2n and an odd integer 2n + 1, with n an integer. Consecutive integers are n,n + 1,n + 2,… . These forms represent all such integers, allowing a general argument rather than a list of examples. Use different letters for independent integers unless they have a stated relationship.

To prove divisibility by k, express the result as k times an integer and explain why the remaining factor is an integer. The final explanatory sentence connects algebra to the claim. For example, 6n + 6 = 6(n + 1) is a multiple of 6 because n + 1 is an integer.

A reliable method

1. Define integer variables and their relationships.

2. Expand or factorise the required expression.

3. Conclude using the definition of evenness, oddness or divisibility.

Worked example 1

Prove the sum of two odd integers is even.

STEP-BY-STEP SOLUTION

Let the integers be 2m + 1 and 2n + 1. Their sum is 2m + 2n + 2 = 2(m + n + 1), twice an integer, so even.

Worked example 2

Prove the sum of three consecutive integers is divisible by 3.

STEP-BY-STEP SOLUTION

n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1). Since n + 1 is an integer, the sum is divisible by 3.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Using 2n + 1 twice proves a statement only for two equal odd integers. For arbitrary odd integers, use independent variables such as m and n.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Prove the sum of two even integers is even.

2.2 Prove an odd integer plus an even integer is odd.

2.3 Prove the difference between consecutive squares n^2 and $(n + 1)^2$ is odd.

2.4 Prove the sum of four consecutive integers is even.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Products, squares and structured arguments — Higher

Some divisibility arguments combine algebra with a property of integers. The product $n(n + 1)$ is even because one of two consecutive integers is even. Factorisation can expose this structure in a more complicated expression. Explaining the integer property is essential; a factorised expression alone may not finish the proof.

For a square of an odd integer, $(2n + 1)^2 = 4n(n + 1) + 1$. Since $n(n + 1)$ is even, the square has form $8k + 1$. Such arguments should state each step and its domain. A universal inequality over real inputs can instead use the fact that a square is non-negative.

A reliable method

1. Simplify to reveal a useful product or square.

2. Explain the relevant integer or non-negative property.

3. State the conclusion in the exact terms of the claim.

Worked example 1

Prove $n^2 + n$ is even for every integer n.

STEP-BY-STEP SOLUTION

$n^2 + n = n(n + 1)$. One of consecutive integers n and $n + 1$ is even, so their product is even.

Worked example 2

Prove $x^2 - 6x + 10$ is positive for every real x.

STEP-BY-STEP SOLUTION

Complete the square: $(x - 3)^2 + 1$. A square is at least zero, so the expression is at least 1 and hence positive.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A proof over integers may rely on parity; a proof over real numbers cannot assume every input is even or odd. State the domain before choosing an argument.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Prove the product of two odd integers is odd.

3.2 Prove $(n + 2)^2 - n^2$ is divisible by 4 for integer n.

3.3 Prove $x^2 + 4x + 7 \geq 3$ for real x.

3.4 Prove the square of an odd integer is one more than a multiple of 8.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Give a counterexample to “the square of every integer is even”. [1 mark]

R2. Prove $2(x + 5) + 3(x - 2)$ is identical to $5x + 4$. [2 marks]

R3. Prove the sum of two odd integers and one even integer is even. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Prove the sum of five consecutive integers is divisible by 5. [3 marks]

R5. Higher: prove $n^2 - n$ is even for integer n. [3 marks]

R6. Higher: prove $x^2 - 2x + 5 \geq 4$ for all real x and state equality. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Conjectures, counterexamples and identities

1.1 Disprove “every odd number is prime”.

9 is odd but composite: $9 = 3 \times 3$.

1.2 Show $4(x + 1) - x$ is identical to $3x + 4$.

Expand: $4x + 4 - x = 3x + 4$.

1.3 Is $x + 2 = 5$ an identity? Explain.

No. It holds only for $x = 3$.

1.4 State the restriction for $x^2/x = x$.

$x \neq 0$ because the original denominator cannot be zero.

Lesson 2 • Parity, consecutive integers and divisibility

2.1 Prove the sum of two even integers is even.

$2m + 2n = 2(m + n)$, twice an integer.

2.2 Prove an odd integer plus an even integer is odd.

$(2m + 1) + 2n = 2(m + n) + 1$, an odd form.

2.3 Prove the difference between consecutive squares n^2 and $(n + 1)^2$ is odd.

$(n + 1)^2 - n^2 = 2n + 1$, which is odd.

2.4 Prove the sum of four consecutive integers is even.

$n + (n + 1) + (n + 2) + (n + 3) = 4n + 6 = 2(2n + 3)$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Products, squares and structured arguments — Higher

3.1 Prove the product of two odd integers is odd.

$(2m + 1)(2n + 1) = 2(2mn + m + n) + 1$, an odd form.

3.2 Prove $(n + 2)^2 - n^2$ is divisible by 4 for integer n.

Expand to $4n + 4 = 4(n + 1)$, a multiple of 4.

3.3 Prove $x^2 + 4x + 7 \geq 3$ for real x.

$x^2 + 4x + 7 = (x + 2)^2 + 3 \geq 3$.

3.4 Prove the square of an odd integer is one more than a multiple of 8.

$(2n + 1)^2 = 4n(n + 1) + 1$. Consecutive product $n(n + 1) = 2k$ for an integer k, giving $8k + 1$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Give a counterexample to “the square of every integer is even”. [1 mark]

$3^2 = 9$ is odd, so 3 is a counterexample. [1 for valid example.]

R2. Prove $2(x + 5) + 3(x - 2)$ is identical to $5x + 4$. [2 marks]

Expand to $2x + 10 + 3x - 6 = 5x + 4$. [1 for expansion; 1 for collection.]

R3. Prove the sum of two odd integers and one even integer is even. [3 marks]

$(2m + 1) + (2n + 1) + 2p = 2(m + n + p + 1)$, twice an integer. [1 for forms; 1 for algebra; 1 for conclusion.]

R4. Prove the sum of five consecutive integers is divisible by 5. [3 marks]

$n + (n + 1) + \ldots + (n + 4) = 5n + 10 = 5(n + 2)$. [1 for representation; 1 for simplification; 1 for integer-factor conclusion.]

R5. Higher: prove $n^2 - n$ is even for integer n. [3 marks]

$n^2 - n = n(n - 1)$. One of these consecutive integers is even, so the product is even. [1 for factorisation; 1 for consecutive parity; 1 for conclusion.]

R6. Higher: prove $x^2 - 2x + 5 \geq 4$ for all real x and state equality. [3 marks]

$x^2 - 2x + 5 = (x - 1)^2 + 4 \geq 4$. Equality occurs at $x = 1$. [1 for completed square; 1 for non-negativity; 1 for equality case.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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