GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 30

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Algebraic reasoning and proof

Shared reasoning with Higher extensions

A proof explains why a statement holds for every permitted case. Use algebra to represent general integers, compare equivalent expressions and construct logical arguments. Learn when a counterexample is enough and when numerical testing is insufficient.

n(n + 1) is even for integer n

A general argument proves more than examples.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Your learning guide

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Before you begin

Expand brackets, factorise expressions and understand integers, parity and divisibility.

Your route through this notebook

1. Conjectures, counterexamples and identities
2. Parity, consecutive integers and divisibility
3. Products, squares and structured arguments — Higher

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Conjectures, counterexamples and identities

A conjecture is a statement that may be true but needs justification. Testing examples can suggest a pattern but does not prove a universal claim. A single counterexample disproves a statement asserting “for all” or “always”. To prove an identity, transform one side into the other using valid algebra, or simplify both sides to the same expression.

An equation can hold for only some inputs, whereas an identity holds for every permitted input. Domain restrictions still matter: $x/x = 1$ is an identity only for $x \neq 0$. Show connected reasoning rather than assuming the desired equality at the start.

A reliable method

1. Identify whether the claim is universal or about a specific input.

2. Use a counterexample to disprove an always-claim.

3. For proof, use general variables and valid transformations.

Worked example 1

Disprove “every prime is odd”.

STEP-BY-STEP SOLUTION

The prime 2 is even, so it is a counterexample.

Worked example 2

Prove $3(x + 2) + 2(x - 1) = 5x + 4$.

STEP-BY-STEP SOLUTION

Expand the left side: $3x + 6 + 2x - 2 = 5x + 4$. This matches the right side for every real x.

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Checking $x = 1$ and $x = 2$ is not proof that two expressions are identical. Different expressions can agree at selected points and differ elsewhere.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Disprove “every odd number is prime”.

1.2 Show $4(x + 1) - x$ is identical to $3x + 4$.

1.3 Is $x + 2 = 5$ an identity? Explain.

1.4 State the restriction for $x^2/x = x$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Parity, consecutive integers and divisibility

An even integer can be written 2n and an odd integer 2n + 1, with n an integer. Consecutive integers are n,n + 1,n + 2,… . These forms represent all such integers, allowing a general argument rather than a list of examples. Use different letters for independent integers unless they have a stated relationship.

To prove divisibility by k, express the result as k times an integer and explain why the remaining factor is an integer. The final explanatory sentence connects algebra to the claim. For example, 6n + 6 = 6(n + 1) is a multiple of 6 because n + 1 is an integer.

A reliable method

1. Define integer variables and their relationships.

2. Expand or factorise the required expression.

3. Conclude using the definition of evenness, oddness or divisibility.

Worked example 1

Prove the sum of two odd integers is even.

STEP-BY-STEP SOLUTION

Let the integers be 2m + 1 and 2n + 1. Their sum is 2m + 2n + 2 = 2(m + n + 1), twice an integer, so even.

Worked example 2

Prove the sum of three consecutive integers is divisible by 3.

STEP-BY-STEP SOLUTION

n + (n + 1) + (n + 2) = 3n + 3 = 3(n + 1). Since n + 1 is an integer, the sum is divisible by 3.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Using 2n + 1 twice proves a statement only for two equal odd integers. For arbitrary odd integers, use independent variables such as m and n.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Prove the sum of two even integers is even.

2.2 Prove an odd integer plus an even integer is odd.

2.3 Prove the difference between consecutive squares n^2 and $(n + 1)^2$ is odd.

2.4 Prove the sum of four consecutive integers is even.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Products, squares and structured arguments — Higher

Some divisibility arguments combine algebra with a property of integers. The product $n(n + 1)$ is even because one of two consecutive integers is even. Factorisation can expose this structure in a more complicated expression. Explaining the integer property is essential; a factorised expression alone may not finish the proof.

For a square of an odd integer, $(2n + 1)^2 = 4n(n + 1) + 1$. Since $n(n + 1)$ is even, the square has form $8k + 1$. Such arguments should state each step and its domain. A universal inequality over real inputs can instead use the fact that a square is non-negative.

A reliable method

1. Simplify to reveal a useful product or square.

2. Explain the relevant integer or non-negative property.

3. State the conclusion in the exact terms of the claim.

Worked example 1

Prove $n^2 + n$ is even for every integer n.

STEP-BY-STEP SOLUTION

$n^2 + n = n(n + 1)$. One of consecutive integers n and $n + 1$ is even, so their product is even.

Worked example 2

Prove $x^2 - 6x + 10$ is positive for every real x.

STEP-BY-STEP SOLUTION

Complete the square: $(x - 3)^2 + 1$. A square is at least zero, so the expression is at least 1 and hence positive.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A proof over integers may rely on parity; a proof over real numbers cannot assume every input is even or odd. State the domain before choosing an argument.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Prove the product of two odd integers is odd.

3.2 Prove $(n + 2)^2 - n^2$ is divisible by 4 for integer n.

3.3 Prove $x^2 + 4x + 7 \geq 3$ for real x.

3.4 Prove the square of an odd integer is one more than a multiple of 8.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Give a counterexample to “the square of every integer is even”. [1 mark]

R2. Prove $2(x + 5) + 3(x - 2)$ is identical to $5x + 4$. [2 marks]

R3. Prove the sum of two odd integers and one even integer is even. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Prove the sum of five consecutive integers is divisible by 5. [3 marks]

R5. Higher: prove $n^2 - n$ is even for integer n. [3 marks]

R6. Higher: prove $x^2 - 2x + 5 \geq 4$ for all real x and state equality. [3 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Conjectures, counterexamples and identities

1.1 Disprove “every odd number is prime”.

9 is odd but composite: $9 = 3 \times 3$.

1.2 Show $4(x + 1) - x$ is identical to $3x + 4$.

Expand: $4x + 4 - x = 3x + 4$.

1.3 Is $x + 2 = 5$ an identity? Explain.

No. It holds only for $x = 3$.

1.4 State the restriction for $x^2/x = x$.

$x \neq 0$ because the original denominator cannot be zero.

Lesson 2 • Parity, consecutive integers and divisibility

2.1 Prove the sum of two even integers is even.

$2m + 2n = 2(m + n)$, twice an integer.

2.2 Prove an odd integer plus an even integer is odd.

$(2m + 1) + 2n = 2(m + n) + 1$, an odd form.

2.3 Prove the difference between consecutive squares n^2 and $(n + 1)^2$ is odd.

$(n + 1)^2 - n^2 = 2n + 1$, which is odd.

2.4 Prove the sum of four consecutive integers is even.

$n + (n + 1) + (n + 2) + (n + 3) = 4n + 6 = 2(2n + 3)$.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Products, squares and structured arguments — Higher

3.1 Prove the product of two odd integers is odd.

$(2m + 1)(2n + 1) = 2(2mn + m + n) + 1$, an odd form.

3.2 Prove $(n + 2)^2 - n^2$ is divisible by 4 for integer n.

Expand to $4n + 4 = 4(n + 1)$, a multiple of 4.

3.3 Prove $x^2 + 4x + 7 \geq 3$ for real x.

$x^2 + 4x + 7 = (x + 2)^2 + 3 \geq 3$.

3.4 Prove the square of an odd integer is one more than a multiple of 8.

$(2n + 1)^2 = 4n(n + 1) + 1$. Consecutive product $n(n + 1) = 2k$ for an integer k, giving $8k + 1$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Give a counterexample to “the square of every integer is even”. [1 mark]

$3^2 = 9$ is odd, so 3 is a counterexample. [1 for valid example.]

R2. Prove $2(x + 5) + 3(x - 2)$ is identical to $5x + 4$. [2 marks]

Expand to $2x + 10 + 3x - 6 = 5x + 4$. [1 for expansion; 1 for collection.]

R3. Prove the sum of two odd integers and one even integer is even. [3 marks]

$(2m + 1) + (2n + 1) + 2p = 2(m + n + p + 1)$, twice an integer. [1 for forms; 1 for algebra; 1 for conclusion.]

R4. Prove the sum of five consecutive integers is divisible by 5. [3 marks]

$n + (n + 1) + \ldots + (n + 4) = 5n + 10 = 5(n + 2)$. [1 for representation; 1 for simplification; 1 for integer-factor conclusion.]

R5. Higher: prove $n^2 - n$ is even for integer n. [3 marks]

$n^2 - n = n(n - 1)$. One of these consecutive integers is even, so the product is even. [1 for factorisation; 1 for consecutive parity; 1 for conclusion.]

R6. Higher: prove $x^2 - 2x + 5 \geq 4$ for all real x and state equality. [3 marks]

$x^2 - 2x + 5 = (x - 1)^2 + 4 \geq 4$. Equality occurs at $x = 1$. [1 for completed square; 1 for non-negativity; 1 for equality case.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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