GCSE ONLINE TUTORING

ALGEBRA & GRAPHS / TOPIC GUIDE 29

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Numerical methods and iteration

Higher-level GCSE material

Some equations are easier to solve approximately than exactly. Learn how a sign change brackets a root, how repeated substitution generates estimates and how to decide whether a requested rounded answer is justified.

$$x_{n+1} = g(x_n)$$

Iterate carefully; use a bracket to justify accuracy.

UNDERSTAND THE IDEA. EXPLAIN THE METHOD.

Detailed teaching • Worked examples • Discussion • Original practice • Full solutions

Name: _____

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Your learning guide

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Before you begin

Substitute into functions, rearrange equations, use roots and retain calculator precision.

Your route through this notebook

1. Locating roots by trial and sign change
2. Repeated substitution and iterative formulae
3. Bisection and proving decimal accuracy

How to use it

Read a lesson, cover the worked solutions and try the examples yourself. Complete its practice page before checking the answers at the back. Finish with the mixed review, then return to any method you could not explain.

YOUR PROGRESS

After each lesson, ask: Can I explain the idea? Can I choose a method? Can I check my result? Record any questions you want to revisit.

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LESSON 1 / UNDERSTAND

Locating roots by trial and sign change

For a continuous function, a change of sign between two inputs guarantees at least one root between them. Calculate the function at the two endpoints and state both the sign change and continuity assumption. This does not prove a unique root unless additional reasoning, such as monotonicity, is available.

A sign change across a discontinuity is not a valid root guarantee: 1/x changes sign across zero but has no root and is undefined at zero. Once a valid interval is found, test points inside it to narrow the bracket. Keep the function definition and every calculated value visible.

A reliable method

1. Rearrange the equation as f(x) = 0.

2. Find endpoint values of opposite signs on a continuous interval.

3. Test intermediate values and retain the smaller sign-changing interval.

Worked example 1

Show a root of $x^3 - x - 2 = 0$ lies between 1 and 2.

STEP-BY-STEP SOLUTION

f(1) = −2 and f(2) = 4. The polynomial is continuous, so a sign change guarantees a root in (1,2).

Worked example 2

Narrow that root using x = 1.5 and x = 1.6.

STEP-BY-STEP SOLUTION

f(1.5) = −0.125 and f(1.6) = 0.496. Therefore a root lies in (1.5,1.6).

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LESSON 1 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

Two positive endpoint values do not prove there is no root between them. A curve could cross the axis twice. A sign-change test is sufficient under continuity, but not necessary for every root.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

1.1 Bracket a root of $x^2 - 5 = 0$ between consecutive positive integers.

1.2 Evaluate $x^3 - 4$ at $x = 1$ and $x = 2$ and interpret.

1.3 Why does the sign change of $1/x$ between -1 and 1 not prove a root?

1.4 For $f(x) = x^2 - 2$, calculate $f(1.4)$ and $f(1.5)$.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 2 / UNDERSTAND

Repeated substitution and iterative formulae

An iterative formula generates a new estimate from the previous one: $x_{n+1} = g(x_n)$. Start from the supplied x_0 and substitute each result back into the same rule. Keep full calculator precision internally, even if the displayed table is rounded. Otherwise rounding can change later estimates.

A rearranged fixed point satisfies $x = g(x)$, but not every rearrangement converges to it. Some iterations oscillate or diverge. If values settle down numerically, they suggest a solution; a residual check in the original equation and a valid bracket give stronger support. Read subscripts carefully: x_0 is the starting value, not the first computed update.

A reliable method

1. Record the starting value with its subscript.

2. Compute each update from the previous unrounded estimate.

3. Check the apparent limit in the original equation.

Worked example 1

Use $x_{n+1} = (x_n + 6/x_n)/2$ with $x_0 = 2$. Find x_1 and x_2.

STEP-BY-STEP SOLUTION

$x_1 = (2 + 3)/2 = 2.5$. $x_2 = (2.5 + 2.4)/2 = 2.45$. This iteration estimates $\sqrt{6}$.

Worked example 2

Rearrange $x^3 - x - 2 = 0$ into a cube-root iteration.

STEP-BY-STEP SOLUTION

$x^3 = x + 2$, so use x_{n+1} = cube root of $(x_n + 2)$. A suitable start near the positive root produces improving estimates.

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LESSON 2 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

A repeated calculator display is evidence of numerical stability at that display precision, not by itself a rigorous proof of the correctly rounded mathematical root.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

2.1 Use $x_{n+1} = (x_n + 4)/2$, $x_0 = 0$, to find x_1, x_2, x_3.

2.2 Find the fixed point of $x_{n+1} = (x_n + 4)/2$.

2.3 Use $x_{n+1} = \sqrt{(x_n + 2)}$, $x_0 = 1$, to find x_1 and x_2 to 4 decimal places.

2.4 For $x_{n+1} = 2x_n$, $x_0 = 1$, list the next four values and describe.

Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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LESSON 3 / UNDERSTAND

Bisection and proving decimal accuracy

Bisection halves a valid sign-changing interval. Test the midpoint, then keep the half whose endpoints still have opposite signs. For a continuous function this retains at least one root. Repeating reduces the interval width predictably, though a uniqueness argument may still be needed for a specific root.

To justify a rounded value, ensure the whole final bracket lies inside that value's rounding interval. For example, an answer 1.52 to 2 decimal places is secure if the root is bracketed between 1.515 and 1.525. Testing 1.52 itself and finding a small residual does not alone establish the rounding.

A reliable method

1. Find the midpoint and calculate its function value.

2. Retain opposite-sign endpoints.

3. Compare final endpoints with rounding boundaries, not just decimal digits.

Worked example 1

Apply one bisection step to $f(x) = x^2 - 2$ on $[1,2]$.

STEP-BY-STEP SOLUTION

Midpoint 1.5 gives $f(1.5) = 0.25$. Since $f(1) = -1$, retain $[1,1.5]$.

Worked example 2

A root is bracketed between 1.521 and 1.522. Give it to 2 decimal places.

STEP-BY-STEP SOLUTION

Both endpoints and every point between round to 1.52, so the root is 1.52 to 2 decimal places.

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LESSON 3 / DISCUSS & PRACTISE

Explain it, then try it

DISCUSSION • TEST THE IDEA

An interval from 1.514 to 1.516 is narrow but crosses the 1.515 rounding boundary. It does not establish a single 2-decimal-place answer. Width alone is not enough.

Say it in your own words

Explain the key idea to someone else. Use a small example to support your explanation. If you disagree with a claim, show why it fails.

3.1 Bisect [2,3] once for $f(x) = x^2 - 5$.

3.2 A root lies in (2.236,2.237). Give it to 2 decimal places.

3.3 A root lies in (4.444,4.446). Is 4.44 guaranteed to 2 decimal places?

3.4 After four bisections, what is the width of an initial interval of width 2?

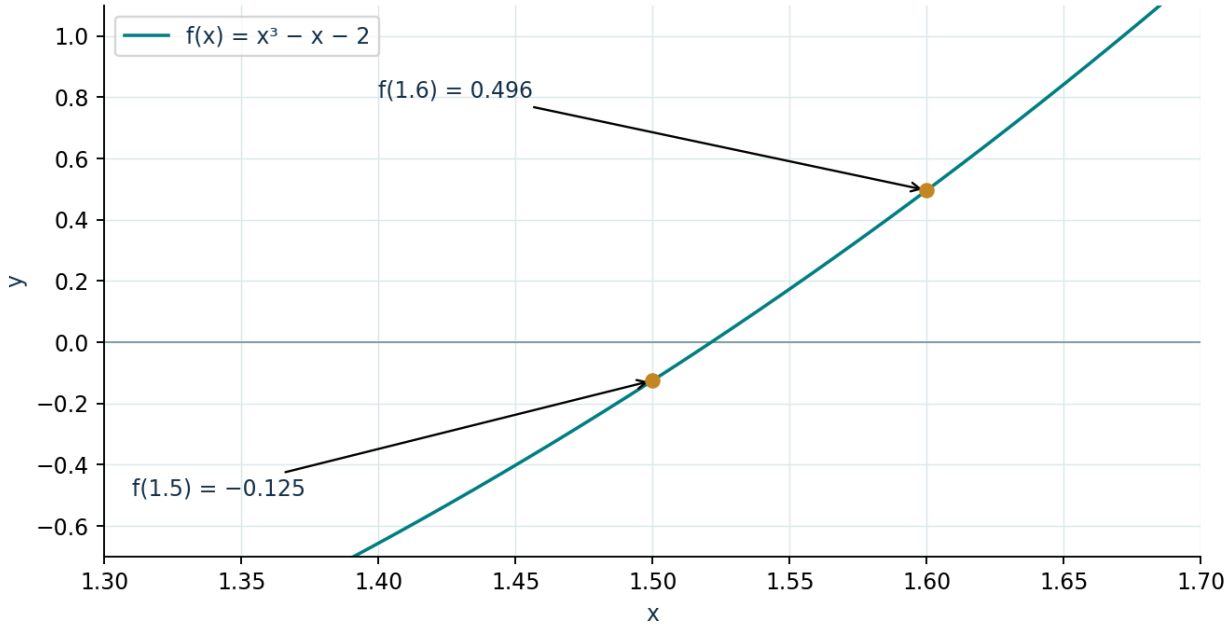
Check before turning to the answers: is the size sensible, is the notation clear, and have you included any required units?

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VISUAL EXPLAINER / CONNECT THE RULE TO THE GRAPH

Read the mathematics

A continuous sign change brackets a root



WHAT THE FIGURE SHOWS

The polynomial is continuous and changes sign between 1.5 and 1.6. A root lies inside that interval. To establish a rounded answer, narrow the bracket until every possible value within it rounds identically; a graph alone usually cannot justify many decimal places.

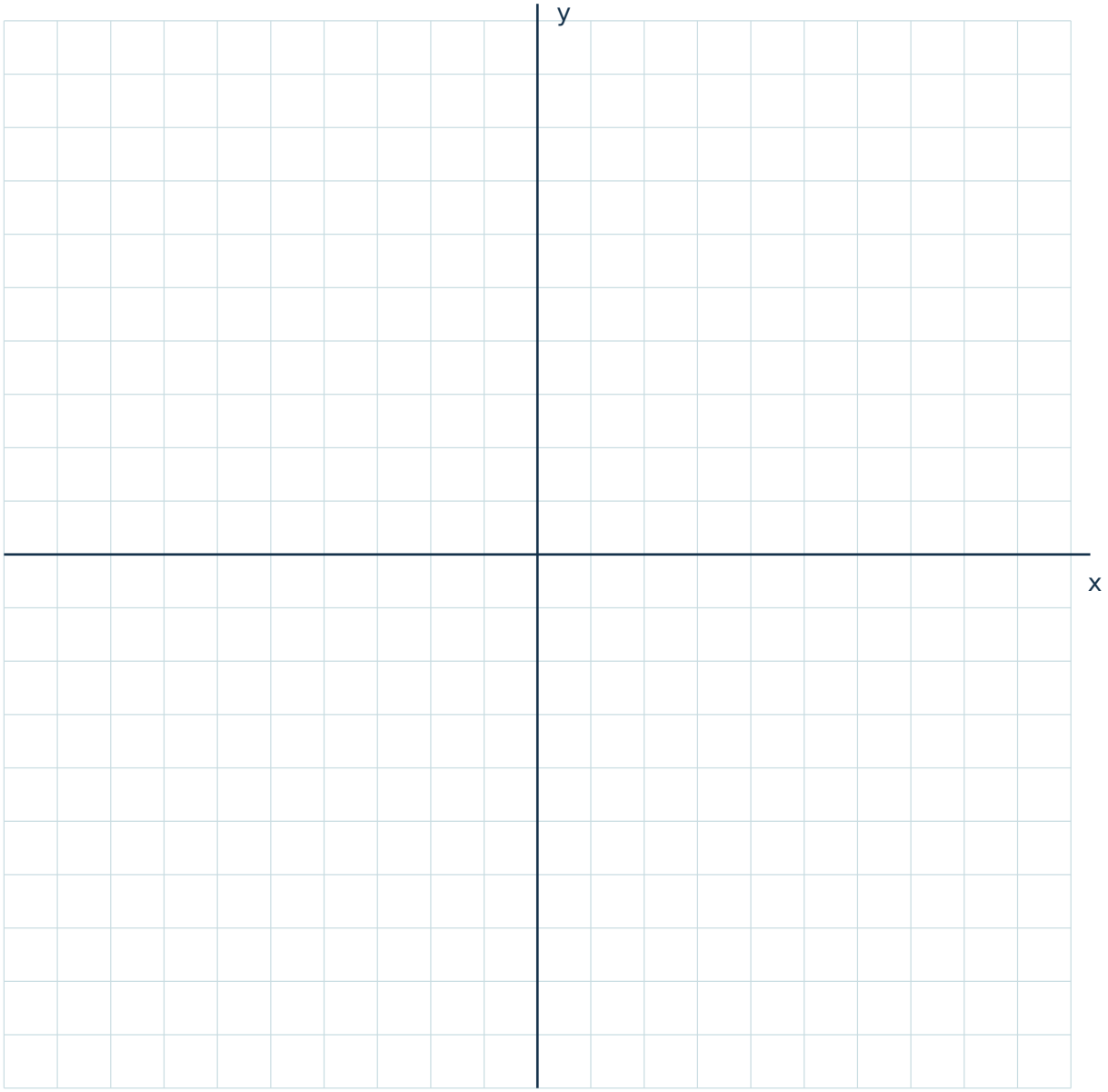
Explain how the labelled features follow from the equations. Check a point by substitution or recalculate a gradient or area before relying on the picture.

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GRAPH WORKSPACE / RECREATE AND EXPLAIN

Build the picture yourself

Choose one graph from the preceding visual explainer and reconstruct it on this grid. Choose and label a suitable scale on each axis; one square need not represent one unit. Include the displayed domain, key points and units. For a real-life graph, relabel x and y with the measured quantities.



SELF-CHECK

Compare with the preceding plotted figure and its explanation. Check at least three points, any intercepts and the direction or shape of the graph. Explain how the equation or context produces those features.

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 1

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R1. Show a root of $x^3 - 5 = 0$ lies between 1 and 2. [3 marks]

R2. For $f(x) = x^3 - 5$, evaluate at 1.7 and 1.8 and narrow the interval. [3 marks]

R3. Use $x_{n+1} = (x_n + 8/x_n)/2$ with $x_0 = 2$. Find x_1, x_2. [3 marks]

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MIXED REVIEW / ORIGINAL EXAM-STYLE PRACTICE

Apply your learning • 2

These questions are independently authored, not taken or adapted from past papers. Marks are suggested for self-assessment. Show your method and explain decisions. Use a calculator only where the arithmetic requires it.

R4. Find the fixed point of $x_{n+1} = (x_n + 10)/3$. [2 marks]

R5. A root lies between 1.709 and 1.710. State it to 2 decimal places and justify. [2 marks]

R6. Explain why $x_{n+1} = 3x_n$ with $x_0 = 1$ does not converge. [2 marks]

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SOLUTIONS / LESSON PRACTICE

Check the reasoning

Your layout may differ. Compare the mathematical reasoning as well as the final answer. Correct any error, then reattempt the question without looking.

Lesson 1 • Locating roots by trial and sign change

1.1 Bracket a root of $x^2 - 5 = 0$ between consecutive positive integers.

$f(2) = -1$ and $f(3) = 4$, so a root lies between 2 and 3.

1.2 Evaluate $x^3 - 4$ at $x = 1$ and $x = 2$ and interpret.

-3 and 4; continuity gives a root between 1 and 2.

1.3 Why does the sign change of $1/x$ between -1 and 1 not prove a root?

The function is not continuous on the interval: it is undefined at 0.

1.4 For $f(x) = x^2 - 2$, calculate $f(1.4)$ and $f(1.5)$.

-0.04 and 0.25, so the positive root lies between 1.4 and 1.5.

Lesson 2 • Repeated substitution and iterative formulae

2.1 Use $x_{n+1} = (x_n + 4)/2$, $x_0 = 0$, to find x_1, x_2, x_3.

2,3,3.5.

2.2 Find the fixed point of $x_{n+1} = (x_n + 4)/2$.

Set $x = (x + 4)/2$: $x = 4$.

2.3 Use $x_{n+1} = \sqrt{(x_n + 2)}$, $x_0 = 1$, to find x_1 and x_2 to 4 decimal places.

$x_1 = \sqrt{3} \approx 1.7321$; $x_2 = \sqrt{(\sqrt{3} + 2)} \approx 1.9319$. Retain unrounded x_1 internally.

2.4 For $x_{n+1} = 2x_n$, $x_0 = 1$, list the next four values and describe.

2,4,8,16. The iteration diverges rather than settling to a finite value.

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SOLUTIONS / LESSON PRACTICE

Check the reasoning • continued

Lesson 3 • Bisection and proving decimal accuracy

3.1 Bisect [2,3] once for $f(x) = x^2 - 5$.

Midpoint 2.5 gives $1.25 > 0$; $f(2) < 0$. Retain [2,2.5].

3.2 A root lies in (2.236,2.237). Give it to 2 decimal places.

2.24.

3.3 A root lies in (4.444,4.446). Is 4.44 guaranteed to 2 decimal places?

No: the interval crosses 4.445, so possible roots round to 4.44 or 4.45.

3.4 After four bisections, what is the width of an initial interval of width 2?

$2/2^4 = 0.125$.

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SOLUTIONS / MIXED REVIEW

Answers and marking guidance

Award the suggested method marks for valid working, even where a later arithmetic slip occurs. Do not count the same step twice. These are independent teaching allocations, not official exam-board mark schemes.

R1. Show a root of $x^3 - 5 = 0$ lies between 1 and 2. [3 marks]

$f(1) = -4$, $f(2) = 3$; the continuous polynomial changes sign. [1 for each value; 1 for continuity/conclusion.]

R2. For $f(x) = x^3 - 5$, evaluate at 1.7 and 1.8 and narrow the interval. [3 marks]

$f(1.7) = -0.087$; $f(1.8) = 0.832$. A root lies in (1.7,1.8). [1 for each value; 1 for interval.]

R3. Use $x_{n+1} = (x_n + 8/x_n)/2$ with $x_0 = 2$. Find x_1,x_2. [3 marks]

$x_1 = 3$; $x_2 = (3 + 8/3)/2 = 17/6 \approx 2.8333$. [1 for first update; 1 for second substitution; 1 for result.]

R4. Find the fixed point of $x_{n+1} = (x_n + 10)/3$. [2 marks]

$3x = x + 10$ gives $x = 5$. [1 for fixed-point equation; 1 for result.]

R5. A root lies between 1.709 and 1.710. State it to 2 decimal places and justify. [2 marks]

1.71, because the entire bracket lies in [1.705,1.715). [1 for value; 1 for rounding interval.]

R6. Explain why $x_{n+1} = 3x_n$ with $x_0 = 1$ does not converge. [2 marks]

The values 1,3,9,27,... grow without bound. [1 for example updates; 1 for divergence.]

Reflect, repair and revisit

Record one secure method, one mistake you corrected and one question to retry. Reattempt the questions you missed in a few days. Explain why your new method works before you compare your answer.

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